

# Yukawa Textures From Heterotic Stability Walls

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## Abstract

A holomorphic vector bundle on a Calabi-Yau threefold,  $X$ , with  $h^{1,1}(X) \geq 2$  can have regions of its Kähler cone where it is slope-stable, that is, where the four-dimensional theory is  $\mathcal{N} = 1$  supersymmetric, bounded by “walls of stability”. On these walls the bundle becomes polystable, decomposing into a direct sum, and the low energy gauge group is enhanced by at least one anomalous  $U(1)$  gauge factor. In this paper, we show that these additional symmetries can strongly constrain the superpotential in the stable region, leading to non-trivial textures of Yukawa interactions and restrictions on allowed masses for vector-like pairs of matter multiplets. The Yukawa textures exhibit a hierarchy; large couplings arise on the stability wall and some suppressed interactions “grow back” off the wall, where the extended  $U(1)$  symmetries are spontaneously broken. A number of explicit examples are presented involving both one and two stability walls, with different decompositions of the bundle structure group. A three family standard-like model with no vector-like pairs is given as an example of a class of  $SU(4)$  bundles that has a naturally heavy third quark/lepton family. Finally, we present the complete set of Yukawa textures that can arise for any holomorphic bundle with one stability wall where the structure group breaks into two factors.

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# 1 Introduction

Compactifications of heterotic string and M-theory [1]-[13] on smooth Calabi-Yau threefolds are an important approach to string phenomenology [14]. In several recent papers [15, 16], the phenomenon of *stability walls* was explored within this context. The idea behind these structures is simple. Heterotic compactifications necessarily involve background gauge fields on the Calabi-Yau space. These are normally chosen so as to preserve  $\mathcal{N} = 1$  supersymmetry in four dimensions. Hence, they must satisfy the the Hermitian Yang-Mills equations with zero slope, equations notoriously difficult to solve. What was shown in [15, 16] is that even if a solution is found in some regions of Kähler moduli space, there is not, in general, a solution in other regions. On the boundary between the “chambers” of Kähler moduli space where supersymmetry is or is not preserved, co-dimension one “stability walls” appear.

On these walls, new Abelian gauge bosons become light and the gauge symmetry of the associated four-dimensional effective theory is enhanced. Although these additional  $U(1)$  symmetries are spontaneously broken in the interior of a supersymmetric region, their effect continues to be felt. In particular, matter fields and some moduli have specific charges under the enhanced symmetries. These charges restrict the form of operators which can appear in the four-dimensional superpotential, not simply on or near the stability wall but, via holomorphy arguments, throughout the entire supersymmetric region. In this paper, we describe the textures in Yukawa couplings that can result from the presence of stability walls in the Kähler cone. We also analyze the constraints these walls can impose on the masses of vector-like pairs of matter multiplets. This is useful for both bottom-up and top-down approaches to phenomenology. From the bottom-up point of view, our analysis will provide a broad and well-defined set of Yukawa and vector-like pair mass textures that can arise naturally in smooth compactifications of heterotic string and M-theory. These textures can be used in model building, and can act as a guide as to what is likely to occur in the heterotic context. In particular, in Appendix A we list all textures that can result from the simplest kinds of stability walls.

From the top-down perspective, our results can also act as an guide to model building. Finding stability wall structure is relatively straightforward within the context of holomorphic vector bundle constructions such as monads [17]-[27], extensions [14, 28, 29, 30] and spectral covers [31]-[35]. In fact, it is simply part of the general analysis to show that a given bundle is somewhere slope-stable, that is, admits a connection obeying the Hermitian Yang-Mills equations. Using our results, this structure provides information about which terms could possibly appear in the associated four-dimensional superpotential. Having access to such information early in the construction of a model

can be extremely useful. Instead of first computing the details of a compactification, calculating the Yukawa couplings and discovering, for example, that the top quark mass vanishes, one can analyze the broad features of the allowed interactions at the start to see if the model has any possibility of being phenomenologically viable.

Green-Schwarz anomalous  $U(1)$  symmetries, and the phenomenological constraints arising from them, have been used extensively in model building in Type II theories (for example, see [37, 38]) and have played an important role in recent work on D-brane instantons [39]–[42]. In addition, such effects have been used to discuss Yukawa textures and hierarchies in F-theory [43, 44]. However, it is important to note that the source of the anomalous  $U(1)$  symmetries in the present work—namely, their origin in the global stability structure of the Kähler cone—is entirely new and provides an interesting contrast to the way that such symmetries arise in other contexts in string theory. It is also worth noting that the Yukawa textures explored in this work are distinct from those previously explored in the heterotic context [45, 46].

For specificity, the explicit examples in this paper involve bundles defined by the monad construction [18]–[22] and by extension [14, 28, 29] over complete intersection Calabi-Yau threefolds [36]. However, our results and conclusions are completely general and apply to any holomorphic vector bundle with Kähler cone sub-structure defined on any Calabi-Yau manifold. The paper is structured as follows. In the next section, we review general heterotic compactifications as well as the mathematics and associated effective field theories of stability walls. In Section 3, we describe the Yukawa textures that can result from the presence of the simplest kind of stability wall. Sections 4 and 5 discuss two generalizations of this; first, to stability walls with more complicated internal structure and, second, to the case where multiple stability walls are present in a single Kähler cone. A phenomenologically interesting example of these ideas is presented in Section 6. Constraints imposed by stability walls on massive vector-like pairs of matter multiplets are analyzed in Section 7. In Section 8, we give our conclusions. The paper has two appendices. Appendix A presents a list of all possible Yukawa textures that can result from the simplest kind of stability walls. In Appendix B, we discuss some technical details associated with the phenomenologically realistic example of Section 6.

## 2 Heterotic Vacua and Vector Bundle Stability

### 2.1 General Definitions

In  $E_8 \times E_8$  heterotic string and M-theory, compactification on a smooth Calabi-Yau threefold is not sufficient to ensure that the four-dimensional effective theory is  $\mathcal{N} = 1$  supersymmetric. Since heterotic compactifications necessarily include background gauge fields, supersymmetry is also dependent on the choice of gauge connection and its properties. Dimensional reduction yields the well-known result that to preserve supersymmetry, these gauge fields must solve the Hermitian

Yang-Mills equations

$$g^{a\bar{b}}F_{a\bar{b}} = 0, \quad F_{ab} = 0, \quad F_{\bar{a}\bar{b}} = 0. \quad (2.1)$$

The latter two equations simply require that the connection be holomorphic. However, the first condition,  $g^{a\bar{b}}F_{a\bar{b}} = 0$ , is a notoriously difficult partial differential equation to solve, involving not only the gauge connection but also the Calabi-Yau metric - an object known only numerically at best [47]-[50]. Fortunately, the Donaldson-Uhlenbeck-Yau theorem [51, 52] presents tractable algebraic conditions under which a solution is guaranteed to exist, without having to construct it explicitly.

The content of the Donaldson-Uhlenbeck-Yau theorem, as relevant in this context, may be stated as follows: *On a compact Kähler manifold, a vector bundle  $V$  admits a connection solving the Hermitian-Yang-Mills equations if and only if  $V$  is a poly-stable holomorphic vector bundle of zero slope.* To explain this statement, we must describe what a poly-stable holomorphic vector bundle is and define the notion of *slope*. The slope of a vector bundle (or sheaf)  $\mathcal{F}$  is given by the integral

$$\mu(\mathcal{F}) = \frac{1}{\text{rk}(\mathcal{F})} \int_X c_1(\mathcal{F}) \wedge J \wedge J, \quad (2.2)$$

where  $X$  is the Calabi-Yau manifold with Kähler form  $J$  and  $c_1(\mathcal{F})$  is the first Chern class of  $\mathcal{F}$ . A vector bundle,  $V$ , is said to be *stable* for a given choice of the Kähler form if every sub-bundle<sup>1</sup> actually defined with respect to  $\mathcal{F}$  in  $V$  with  $\text{rk}(\mathcal{F}) < \text{rk}(V)$  has slope strictly less than that of the bundle itself. That is,

$$\mu(\mathcal{F}) < \mu(V) \quad \forall \mathcal{F} \text{ in } V. \quad (2.3)$$

A bundle is called *semi-stable* if  $\mu(\mathcal{F}) \leq \mu(V)$  for all proper sub-bundles  $\mathcal{F}$ . We note that it is not stability that appears in the statement of the Donaldson-Uhlenbeck-Yau theorem, but poly-stability. A bundle is *poly-stable* if it is a direct sum of stable bundles, all of which have the same slope. That is,

$$V = \bigoplus_i V_i \quad \mu(V) = \mu(V_i) \quad \forall i. \quad (2.4)$$

Clearly, all poly-stable bundles are semi-stable, but the converse does not hold. Hence, semi-stable bundles will be of interest to us only when they are also poly-stable.

An essential property, both mathematically and for physical applications, of the notion of stability - as well as semi-stability and poly-stability - is that it depends explicitly on the choice of Kähler form  $J$  on  $X$ . To understand the exact meaning of this, it is useful to expand  $J$  in a basis  $J_i$ ,  $i = 1, \dots, h^{1,1}(X)$ , of  $(1,1)$ -forms as  $J = t^i J_i$ . The coefficients  $t^i$  are the Kähler moduli. Inserting this into (2.2), the slope of any sub-bundle  $\mathcal{F}$  can be written as

$$\mu(\mathcal{F}) = \frac{1}{\text{rk}(\mathcal{F})} d_{ijk} c_1^i(\mathcal{F}) t^j t^k, \quad (2.5)$$

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<sup>1</sup>Really a subsheaf. Stability of a vector bundle is defined so that  $\mu(\mathcal{F}) < \mu(V)$  for all torsion-free sub-sheaves,  $\mathcal{F} \in V$  with  $\text{rk}(\mathcal{F}) < \text{rk}(V)$ . However, for the examples in this work, all the de-stabilizing sub-objects will be bundles and hence for simplicity we will not discuss sheaves.

where  $d_{ijk} = \int_X J_i \wedge J_j \wedge J_k$  are the triple intersection numbers of  $X$  and  $c_1(\mathcal{F}) = c_1^i(\mathcal{F})J_i$ . That is, the slope of each sub-bundle  $\mathcal{F}$  is a calculable function of the Kähler moduli  $t^i$ . It follows that whether or not a bundle is stable, poly-stable or semi-stable is, in general, a function of where one is in Kähler moduli space. *A vector bundle  $V$  which is stable in one region of the Kähler cone of  $X$  may not necessarily be stable in another.*

## 2.2 Stability Walls and Kähler Cone Substructure

How does one determine the the regions of stability/instability of a vector bundle? We begin by noting that the stability properties of a vector bundle for a choice of Kähler class<sup>2</sup>  $J$  will remain unchanged if that Kähler class is multiplied by a non-vanishing complex number. Hence, the stability properties of a bundle are the same along any one-dimensional ray in the Kähler cone. It follows that for a Calabi-Yau manifold with  $h^{1,1}(X) = 1$ , a vector bundle will either be stable, or unstable, everywhere in the one-dimensional Kähler cone. We will, therefore, restrict our discussion to Calabi-Yau threefolds,  $X$ , with  $h^{1,1}(X) \geq 2$ . Now consider a holomorphic vector bundle,  $V$ , on  $X$  such that for at least one choice of Kähler form - and, hence, for the ray it defines - the bundle is slope stable. In this paper, we take all slope-stable bundles to be indecomposable<sup>3</sup> with structure group  $SU(n)$ . Hence, the first Chern class satisfies  $c_1(V) = 0$ . It follows from (2.2) that the slope of  $V$  also vanishes. Thus, for an  $SU(n)$  bundle to be stable for a given value of the Kähler moduli, the slope of each of its sub-bundles, calculated using the corresponding Kähler form  $J$ , must be negative.

It is quite possible to find bundles for which the slopes of all sub-bundles remain negative everywhere in the Kähler cone (for example, the tangent bundle,  $TX$ , to the Calabi-Yau threefold). Any such vector bundle will admit an  $SU(n)$  connection satisfying the Hermitian Yang-Mills equations for any values of the Kähler moduli. Now, however, consider a case where there is one particular sub-bundle  $\mathcal{F}$  (itself stable) whose slope, while negative for the polarization where the bundle  $V$  is assumed stable, gets smaller and smaller as one moves in the Kähler cone, eventually going to zero. The condition  $\mu(\mathcal{F}) = 0$  is one equation restricting  $h^{1,1}$  Kähler moduli. That is, the vanishing of the slope of  $\mathcal{F}$  defines a co-dimension one boundary - called a “stability wall” - in the Kähler cone. As we cross this wall, this sub-bundle becomes positive in slope and destabilizes the vector bundle. That is, the bundle no longer satisfies the Hermitian Yang-Mills equations and supersymmetry is broken. For such bundles, the Kähler cone has sub-structure [15, 16, 53]; that is, it can split into separate “chambers” with respect to the stability properties of  $V$ . In one of these chambers a solution to (2.1) can be found, and in the others it can not. This stability induced sub-structure, and the effective field theory [15, 16] description of it, will be of central importance to this work.

On the boundary between a supersymmetric and non-supersymmetric chamber of the Kähler cone, we know from the proceeding discussion that there is a sub-bundle  $\mathcal{F}$  injecting into the bundle

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<sup>2</sup>The choice of a Kähler form  $J$ , is referred to as a “polarization” in the mathematics literature.

<sup>3</sup>Note that decomposable vector bundles  $V = \bigoplus_i V_i$  can at best be best poly-stable, see (2.4).

$V$  which has the same slope as the bundle itself. That is, we can write an injective morphism  $0 \rightarrow \mathcal{F} \rightarrow V \rightarrow \dots$ . Coherent sheaves form an Abelian category and, thus, one may always write a cokernel,  $\mathcal{K} = V/\mathcal{F}$ , to form a short exact sequence and re-express the bundle as the extension

$$0 \rightarrow \mathcal{F} \rightarrow V \rightarrow \mathcal{K} \rightarrow 0 . \quad (2.6)$$

In other words, no matter how the bundle was originally defined, if it has a stability wall then it may be written as an extension<sup>4</sup>.

Given that on the stability wall  $\mathcal{F}$  injects into  $V$  and has equal slope, the only way in which  $V$  can preserve supersymmetry, according to the Donaldson-Uhlenbeck-Yau theorem, is if it splits into a direct sum of two pieces. In other words, supersymmetry is only preserved on the wall when the sequence (2.6) splits and

$$V = \mathcal{F} \oplus \mathcal{K} . \quad (2.7)$$

Is this always possible? To answer this, note that the set of equivalent extensions,  $V$ , in (2.6) is described by the group  $\text{Ext}^1(\mathcal{K}, \mathcal{F})$ . The split configuration, (2.7), corresponds precisely to the zero element in that space [16]. Thus, as we approach a stability wall in Kähler moduli space, the system can continue to preserve  $\mathcal{N} = 1$  supersymmetry<sup>5</sup>. The price for this, however, is a decomposition of the bundle into two pieces  $V = \mathcal{F} \oplus \mathcal{K}$ . Such a splitting of the bundle on the stability wall corresponds physically to a change in the group in which the gauge field background of the compactification is valued. If we begin with a stable  $SU(n)$  bundle  $V$  then, at the stability wall, the structure group changes to  $S[U(n_1) \times U(n - n_1)]$  where  $n_1$  is the rank of  $\mathcal{F}$ . The exact splitting depends on the choice of structure group  $SU(n)$  in the stable chamber and exactly which sub-bundle destabilizes the bundle at the stability wall. Generically, however, we can see that the effect of the splitting (2.7) will be to change the low-energy effective theory associated with this compactification. If we denote the commutant within  $E_8$  of  $SU(n)$  as  $H$  - the symmetry group of the four-dimensional theory in the stable chamber - then the commutant of  $S[U(n_1) \times U(n - n_1)]$  will be enhanced by one additional anomalous gauged  $U(1)$  symmetry to  $H \times U(1)$ .

## 2.3 Example: An $SU(3)$ Heterotic Compactification

To give a concrete example of such a compactification, consider the Calabi-Yau threefold defined by a bi-cubic polynomial in  $\mathbb{P}^2 \times \mathbb{P}^2$

$$\left[ \begin{array}{c|c} \mathbb{P}^2 & 3 \\ \mathbb{P}^2 & 3 \end{array} \right]^{2,83} , \quad (2.8)$$

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<sup>4</sup>Strictly speaking, this is true if the bundle has a stability wall caused by a single destabilizing sub-bundle. We will discuss more general cases in later sections.

<sup>5</sup>Mathematically, this statement can be understood by saying that on the wall, the semi-stable bundle,  $V$ , is an element of an S-equivalence class [54]. Since each S-equivalence class contains a unique poly-stable representative, it is always possible for the bundle to decompose as in (2.7).

where the superscripts are  $h^{1,1}$  and  $h^{1,2}$  respectively. On this manifold, let us define a holomorphic vector bundle,  $V$ , with structure group  $SU(3)$ . The bundle is given by a two-step process. First construct a rank 2 bundle,  $\mathcal{G}$ , by the so-called monad construction [18]- [22] via the short exact sequence

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}(1,0)^{\oplus 3} \oplus \mathcal{O}(1,1) \xrightarrow{f} \mathcal{O}(1,2) \oplus \mathcal{O}(2,2) \rightarrow 0 . \quad (2.9)$$

$\mathcal{G}$  is defined in terms of the bundle morphism,  $f$ , above as  $\mathcal{G} = \ker(f)$ . Next, we proceed to build the rank three bundle,  $V$ , out of the line bundle  $\mathcal{O}(-1,3)$  and  $\mathcal{G}$ , by “extension”. That is,

$$0 \rightarrow \mathcal{O}(-1,3) \rightarrow V \rightarrow \mathcal{G} \rightarrow 0 . \quad (2.10)$$

The manifold (2.8) is a complete intersection Calabi-Yau manifold [36]. There are only two independent harmonic  $(1,1)$  forms on  $X$ , and a basis  $J_1, J_2$  may be chosen which are the restrictions of the Kähler forms of each  $\mathbb{P}^2$  to the Calabi-Yau hypersurface. In (2.9) and (2.10) above,  $\mathcal{O}(k,m)$  denotes a line bundle on  $X$ . The pair of integers  $(k,m)$  fully specify the line bundle on  $X$  by defining its first Chern class,  $c_1(L) = kJ_1 + mJ_2$ .

The extension bundle  $V$  in (2.10) can be viewed as a non-trivial deformation of the direct sum of the two pieces,  $\mathcal{G}$  and  $\mathcal{O}(-1,3)$ . At a generic point in its moduli space, that is, for generic choices of the maps in (2.9) and (2.10), the bundle  $V$  has structure group  $SU(3)$ . For some choices of Kähler form, it is slope stable and, hence,  $V$  corresponds to an  $SU(3)$  valued solution to (2.1). To find where  $V$  is stable, one must find all sub-bundles  $\mathcal{F}$ , calculate their slopes and check that these are all negative. For such an analysis, see, for example, [16, 23]. Here, we simply present our results.

Figure 1 shows the two-dimensional Kähler cone of Calabi-Yau threefold (2.8). The physical Kähler cone, where the Calabi-Yau is positive in volume and non-singular, is the complete colored region. The light blue, upper region, in Figure 1 is the set of polarizations for which the slope of each sub-bundle of the bundle is negative and, hence, the bundle is stable.

Now note that the description of bundle (2.10) is already in the form (2.6). We can, therefore, simply read off  $\mathcal{F}$  and  $\mathcal{K}$  from (2.6) as

$$\mathcal{F} = \mathcal{O}(-1,3) , \quad (2.11)$$

$$\mathcal{K} = \mathcal{G} \quad \text{where} \quad 0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}(1,0)^{\oplus 3} \oplus \mathcal{O}(1,1) \rightarrow \mathcal{O}(1,2) \oplus \mathcal{O}(2,2) \rightarrow 0 . \quad (2.12)$$

It follows that the bundle  $V$  in (2.10) has a stability wall of the kind we have been describing. This wall is shown as the line in Figure 1. It separates the region of stability of  $V$  from its region of instability. The splitting  $V \rightarrow \mathcal{F} \oplus \mathcal{K}$  on the stability wall corresponds physically to a change in the group in which the gauge field background is valued. For this example, these gauge fields change from being valued in  $SU(3)$  in the interior of the supersymmetric region, to being valued in  $S[U(2) \times U(1)] \cong SU(2) \times U(1)$  on the stability wall. Recall that the four-dimensional symmetry group is the commutant of the structure group of the bundle inside  $E_8$ . Thus, while the low-energy gauge group is simply  $E_6$  in the stable region, an extra Abelian factor appears when the moduli are exactly on the stability wall. Here, the effective gauge symmetry is enhanced to  $E_6 \times U(1)$ .



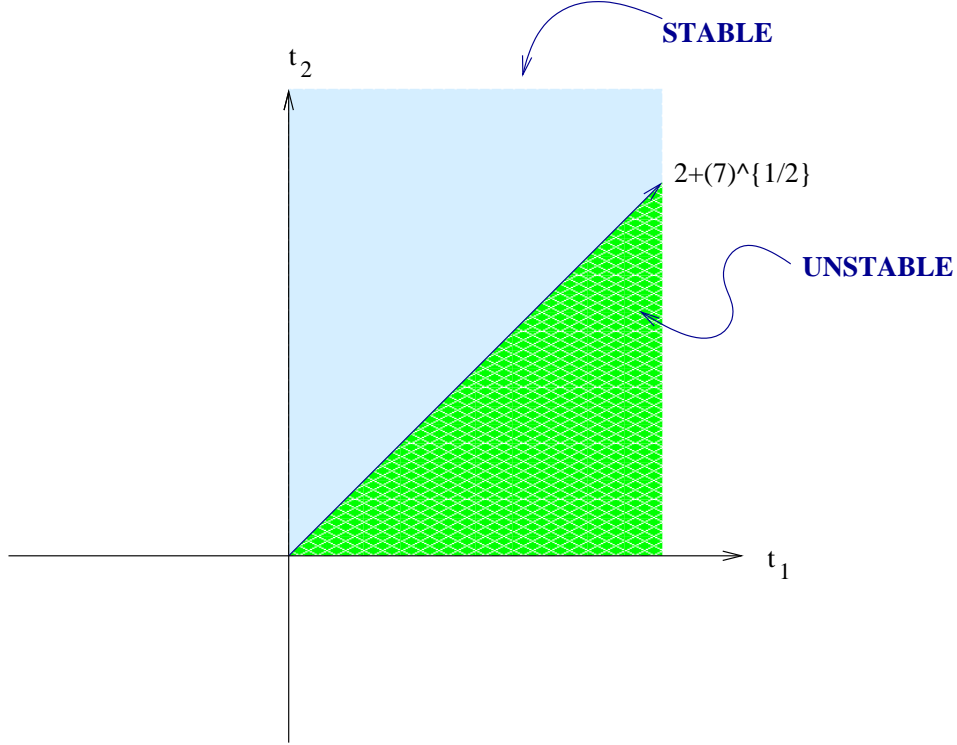


Figure 1: The Kähler cone and regions of stability/instability for Calabi-Yau threefold (2.8) and the bundle (2.10). The stability wall generated by  $\mathcal{O}(-1, 3)$  in  $V$  occurs on the line with slope  $t^2/t^1 = 2 + \sqrt{7}$ .

## 2.4 The Particle Spectrum and Quantum Numbers

An analysis of the particle spectrum and the associated quantum numbers, both in the interior of the stable region of the Kähler cone as well as on a stability wall, is most easily presented in the context of an explicit example. Let us use the Calabi-Yau threefold,  $X$ , and the  $SU(3)$  vector bundle presented in (2.8) and (2.9),(2.10) above.

In the interior of the stable region, the background gauge fields have structure group  $SU(3)$  and the symmetry group of the the four-dimensional effective field theory is  $E_6$ . Computing the matter spectrum of this low energy theory is an exercise in group theory and bundle cohomology [2]. All matter fields in the ten-dimensional theory are valued in the **248** representation of  $E_8$ . The matter multiplets that appear in the four-dimensional spectrum are determined by the branching of this representation under

$$E_8 \supset E_6 \times SU(3) \quad (2.13)$$

$$\mathbf{248} = (\mathbf{78}, \mathbf{1}) + (\mathbf{8}, \mathbf{1}) + (\mathbf{27}, \mathbf{3}) + (\overline{\mathbf{27}}, \overline{\mathbf{3}}) . \quad (2.14)$$

The first number in the brackets above is the dimension of a representation of  $E_6$  and the second the dimension of a representation of  $SU(3)$ . To find the multiplicity of each term, one must compute the number of zero-modes of the associated twisted Dirac operators on the internal space [2]. This is given by the dimension of the relevant bundle-valued cohomology group. The group

Representation	Field Name	Cohomology	Multiplicity
$(\mathbf{8}, \mathbf{1})$	$\phi$	$h^1(X, V \otimes V^*)$	87
$(\mathbf{27}, \mathbf{3})$	$F^I$	$h^1(X, V)$	39
$(\overline{\mathbf{27}}, \overline{\mathbf{3}})$	$\overline{F}^A$	$h^1(X, V^*)$	0

Table 1: The representations, field content and the associated cohomologies for a *generic*  $E_6$  theory. For a Calabi-Yau threefold  $h^1(X, \mathcal{O}_X) = 0$  and  $n_{78} = h^0(X, \mathcal{O}_X) = 1$ . The multiplicities for the *specific* indecomposable rank 3 vector bundle  $V$  defined in (2.10) are given in the fourth column.

representations, four-dimensional field names and the associated cohomologies for a *generic*  $E_6$  theory are indicated in the first three columns of Table 1.

The dimensions of the cohomologies for the *specific* bundle  $V$  in example (2.10) are presented in the fourth column. We see, in particular, that we have 39  $\mathbf{27}$  dimensional representations of  $E_6$ . At this stage, there is nothing to suggest any sort of “texture” in the cubic self-interactions of these fields. Generically, one would expect all Yukawa terms which are allowed by  $E_6$  gauge symmetry to appear. In fact, this is *not* the case, as we will show in the next section.

For a Kähler form *on the stability wall*, the background gauge fields are valued in  $S[U(2) \times U(1)] \cong SU(2) \times U(1)$  and the symmetry group of the four-dimensional theory is  $E_6 \times U(1)$ . The method for computing the spectrum and quantum numbers on the stability wall is analogous to the procedure above. The only difference is that one now takes the gauge bundle to be  $V = \mathcal{F} \oplus \mathcal{K}$ , rather than indecomposable and rank 3. The group theory which determines which multiplets can appear in four dimensions is now

$$E_8 \supset E_6 \times SU(2) \times U(1) \quad (2.15)$$

$$\begin{aligned} \mathbf{248} = & (\mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{2})_3 + (\mathbf{1}, \mathbf{2})_{-3} + (\mathbf{1}, \mathbf{3})_0 + (\mathbf{78}, \mathbf{1})_0 \\ & + (\mathbf{27}, \mathbf{1})_{-2} + (\mathbf{27}, \mathbf{2})_1 + (\overline{\mathbf{27}}, \mathbf{1})_2 + (\overline{\mathbf{27}}, \mathbf{2})_{-1} . \end{aligned} \quad (2.16)$$

Note that each multiplet has an additional quantum number associated with the  $U(1)$  factor in the effective theory. The group representations, four-dimensional field names and the associated cohomologies for a *generic*  $E_6 \times U(1)$  theory are indicated in the first three columns of Table 2. The multiplicity is found by calculating the dimension of each cohomology. The results for the decomposition  $\mathcal{F} \oplus \mathcal{K}$  associated with the *explicit* example (2.11),(2.12) are given in the fourth column. It is important to note here that the extra  $U(1)$  symmetry is Green-Schwarz anomalous, as described in detail in [15, 16]. Thus, the usual anomaly cancellation constraints on the charges do not apply. For the general form of  $U(1)$  charges possible in the present context, see [55]. We will come back to the anomalous nature of this  $U(1)$  in the following sections.

One obvious question is: what is the relationship between the particle spectrum on the stability wall, given in Table 2, and the manifestly different spectrum in the interior of the stability region, presented in Table 1? Furthermore, how does one relate their two four-dimensional field theories?

Representation	Field Name	Cohomology	Multiplicity
$(\mathbf{1}, \mathbf{2})_3$	$C_1^p$	$h^1(X, \mathcal{F}^* \otimes \mathcal{K})$	0
$(\mathbf{1}, \mathbf{2})_{-3}$	$C_2^Q$	$h^1(X, \mathcal{F} \otimes \mathcal{K}^*)$	21
$(\mathbf{1}, \mathbf{3})_0$	$\psi$	$h^1(X, \mathcal{K} \otimes \mathcal{K}^*)$	67
$(\mathbf{27}, \mathbf{1})_{-2}$	$F_1^i$	$h^1(X, \mathcal{F})$	3
$(\mathbf{27}, \mathbf{2})_1$	$F_2^\eta$	$h^1(X, \mathcal{K})$	36
$(\overline{\mathbf{27}}, \overline{\mathbf{1}})_2$	$\overline{F}_1^a$	$h^1(X, \mathcal{F}^*)$	0
$(\overline{\mathbf{27}}, \overline{\mathbf{2}})_{-1}$	$\overline{F}_2^\alpha$	$h^1(X, \mathcal{K}^*)$	0

Table 2: The representations, field content and the cohomologies of a *generic*  $E_6 \times U(1)$  theory associated with a poly-stable bundle  $\mathcal{F} \oplus \mathcal{K}$  on the stability wall. Note that  $h^1(X, \mathcal{F} \otimes \mathcal{F}^*)$  vanishes here since  $\mathcal{F}$  is a line bundle. The multiplicities for the *explicit* bundle defined by (2.7) and (2.11),(2.12) are shown in the fourth column.

To answer these questions, we construct the effective theory on the stability wall and then consider small perturbations into the interior of the slope-stable region.

## 2.5 Connecting the Two Theories

The effective theories associated with the stable bundle  $V$  and the poly-stable bundle  $\mathcal{F} \oplus \mathcal{K}$ , described generically in Section 2.2, can be related by considering the vacuum near the stability wall. This relationship is most easily illustrated using the specific example in Subsection 2.3. Begin with the Kähler moduli of the  $E_6 \times U(1)$  theory on the stability wall in Figure 1. Then vary them continuously, moving away from the boundary and into the stable region of the Kähler cone. This should reproduce the physics of the  $E_6$  compactification.

As shown in [15, 16], the effective theory both on and near the stability wall is described by a D-term associated with the enhanced gauged  $U(1)$  factor. It is given by

$$D^{U(1)} = \frac{3}{16} \frac{\epsilon_S \epsilon_R^2}{\kappa_4^2} \frac{\mu(\mathcal{F})}{\mathcal{V}} - \frac{1}{2} \sum_{P, \overline{Q}} Q_2 G_{P\overline{Q}} C_2^P \overline{C}_2^{\overline{Q}}, \quad (2.17)$$

where the charge<sup>6</sup>  $Q_2 = -3$ . The first term is a Kähler modulus dependent ‘‘Fayet-Iliopoulos’’ (FI) term. This is a multiple of the slope of the destabilizing sub-bundle, divided by the volume  $\mathcal{V}$  of the Calabi-Yau threefold. The constants  $\epsilon_S$  and  $\epsilon_R$  are the usual expansion parameters defining four-dimensional heterotic M-theory [8]. It follows from the discussion in Subsection 2.3 that the FI term is positive in the non-supersymmetric (dark shaded) region of Figure 1, negative in the

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<sup>6</sup>Note that locally  $S[U(2) \times U(1)] \approx SU(2) \times U(1)$ . Globally, however, there is different overall normalization on the  $U(1)$  which commutes with this group within  $E_8$ . In Ref. [16] the  $U(1)$  normalization was chosen consistent with the global description. In this work, since we are interested only in gauge invariant quantities (where overall normalization does not matter) we have chosen for simplicity the *local* charge normalizations consistent with Ref. [57].

stable (light shaded) region, and vanishes on the boundary line between the two. The second term is the usual contribution to a D-term from charged matter. The fields shown in (2.17) are the  $E_6$  singlets  $C_2$  in Table 2. The positive definite field space metric  $G_{P\bar{Q}}$  appears since they are not generically canonically normalized. In the explicit example of Subsection 2.3, there are no  $C_1$  fields in the spectrum. The 27 and  $\bar{27}$  representations in Table 2, which are also charged under  $U(1)$ , should appear in this D-term as well. However, the  $E_6$  D-terms force the vevs of these fields to vanish, and hence, they can be safely ignored in the following discussion.

Using the D-term (2.17), one can concretely specify the relationship between the effective theories in the stable, poly-stable and unstable regions of Figure 1. On the stability wall,  $\mu(\mathcal{F}) = 0$  and the  $D^{U(1)}$  contribution to the potential is minimized for  $\langle C_2^P \rangle = 0$ . Hence, the theory is  $E_6 \times U(1)$  invariant with the spectrum of massless fields given in Table 2. Strictly speaking, the  $U(1)$  factor is Green-Schwarz anomalous. Hence, the associated gauge boson is not massless, even on the stability wall. The mass of this gauge boson was computed in [15, 16, 55, 56]. On the stability wall it was found to be

$$m_{U(1)}^2 = \frac{1}{s} \left( \frac{(3\epsilon_S \epsilon_R^2)^2}{256\kappa_4^2} c_1^i(\mathcal{F}) c_1^j(\mathcal{F}) G_{ij} \right), \quad (2.18)$$

where  $G_{ij} = -\frac{\partial^2 \ln \mathcal{V}}{\partial v^i \partial v^j}$  and  $s = \text{Re} S$  is the real part of the dilaton. This is parametrically lighter than the compactification scale and, hence, the Abelian gauge boson must be included in the four-dimensional effective theory. What happens to D-term (2.17) as one moves continuously off the stability wall and into the stable region of moduli space? Here,  $\mu(\mathcal{F}) < 0$  and the  $C_2$  fields acquire non-zero vevs so as to set  $D^{U(1)} = 0$  and minimize the potential. The  $\langle C_2^P \rangle \neq 0$  vevs thus spontaneously break  $U(1)$ , reducing the symmetry to a pure  $E_6$  gauge theory. Specifically, the mass of the  $U(1)$  gauge boson is enhanced [15, 16] from (2.18) to

$$m_{U(1)}^2 = \frac{1}{s} \left( \frac{(3\epsilon_S \epsilon_R^2)^2}{256\kappa_4^2} c_1^i(\mathcal{F}) c_1^j(\mathcal{F}) G_{ij} + \frac{9}{4} \sum_{P,\bar{Q}} G_{P\bar{Q}} \langle C_2 \rangle^P \langle \bar{C}_2 \rangle^{\bar{Q}} \right). \quad (2.19)$$

As  $\langle C_2 \rangle$  increases in magnitude, their contribution drives the  $U(1)$  gauge boson mass above the compactification scale. It must then be integrated out and removed from the four-dimensional theory. This process, both on and off of the stability wall, is simply a Higgs effect.

Expanding each field as a small fluctuation around its vev and using  $\langle D^{U(1)} \rangle = 0$ , the D-term (2.17) is given to linear order by

$$\delta D^{U(1)} = -\frac{3}{16} \frac{\epsilon_S \epsilon_R^2}{\kappa_4^2} G_{jk} c_1^j(\mathcal{F}) \delta t^k - \frac{3}{2} \sum_{P,\bar{Q}} G_{P\bar{Q}} \left( \langle C_2^P \rangle \delta \bar{C}_2^{\bar{Q}} + \delta C_2^P \langle \bar{C}_2^{\bar{Q}} \rangle \right). \quad (2.20)$$

From  $V = \frac{1}{2s} (\delta D^{U(1)})^2$ , canonically normalizing the kinetic energy and using (2.19), one can extract that massive Higgs field as a linear combination of  $\delta t^k$  and  $\delta C_2^P$  fluctuations. This is explicitly discussed in [15, 16]. Suffice it here to say that on the stability wall, the Higgs field reduces to the linear combination of  $\delta t^1, \delta t^2$  perpendicular to the line in Figure 1. The associated linear

combination of Kähler moduli axions acts as the Goldstone boson and is “eaten” so as to give additional mass to the  $U(1)$  gauge boson. Thus, near the stability wall one entire complex linear combination of Kähler moduli becomes heavy due to the Higgs mechanism. As one moves away from the stability wall in Kähler moduli space, the vevs of the  $C_2$  fields adjust so as to minimize the potential. As discussed in [15, 16], the  $\delta C_2^P$  terms quickly become the dominant contribution to the Higgs field. Thus, in the stable region far from the wall, essentially one complex  $C_2$  field is lost to the Higgs effect.

One can now explicitly describe the transition from the massless  $E_6 \times U(1)$  spectrum on the stability wall, given in Table 2, to the  $E_6$  zero-mode spectrum in the interior of the stable region, Table 1. Of the 21  $C_2$  fields on the stability wall, 1 of them is lost through the Higgs mechanism as one moves into the stable region. Integrating out the heavy  $U(1)$  gauge boson, the -3 charge of the remaining 20  $C_2$  fields can be ignored. These combine with the 67  $\psi$  fields of Table 2 to correctly reproduce the 87 uncharged bundle moduli of the  $E_6$  theory in Table 1. Furthermore, when the  $U(1)$  symmetry is integrated out, the quantum numbers distinguishing the two types of **27** fields at the stability wall, 3 with  $U(1)$  charge -2 and 36 with charge +1, no longer label the spectrum. Thus, we find the expected 39 **27** fields of Table 1. This correspondence between the massless spectrum near a stability wall and that in the interior of the stable region was proven in complete generality in [16]<sup>7</sup>.

Finally, let us start once again on the stability wall. Now, continuously vary the moduli into the unstable region, where  $\mu(\mathcal{F}) > 0$ . In principle, the  $C_1$  fields with  $U(1)$  charge  $Q_1 = +3$  could cancel the positive FI-term. However, for the bundle in (2.11) and (2.12), we see from Table 2 that there are no  $C_1$  fields present in the spectrum. Therefore  $D^{U(1)} \neq 0$  and supersymmetry is broken, as we expect from the stability analysis.

## 2.6 The Charged Bundle Moduli $C_i$ and Branch Structure

In this subsection, for specificity, we consider rank three bundles whose Kähler cone contains at least one stability wall. Furthermore, our analysis is confined to a single wall where the  $SU(3)$  structure group decomposes into  $S[U(2) \times U(1)]$ . Hence, the  $E_6$  gauge group is enhanced by a single  $U(1)$  factor, giving rise to one Abelian D-term in the effective theory. Our discussion will, therefore, be applicable to the specific example discussed in Subsections 2.3, 2.4 and 2.5, but will be considerably more general. We emphasize that the type of conclusions drawn from this analysis will remain unchanged for bundles of higher rank, and for stability walls described by more than one D-term.

Consider a general rank three bundle  $V$ , destabilized by a single sub-bundle  $\mathcal{F}$  as in (2.6),

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<sup>7</sup>Note that in moving between the stable region and the poly-stable wall, only the chiral asymmetry need be preserved. The actual number of 27 and  $\overline{27}$  representations does not necessarily remain the same. In particular, massless vector-like pairs on the stability wall can become massive in the stable region of moduli space. We will return to the issue of massive vector-like pairs, and possible constraints on them, in Section 7.

which generates Kähler cone sub-structure of the form discussed in Section 2.2. In general, for such a bundle, there are precisely two types of bundle moduli *charged under the extended  $U(1)$  symmetry*. These are denoted  $C_1, C_2$  and arise from the cohomologies shown in Table 2. These charged bundle moduli, by acquiring vevs to cancel the FI-term, play a central role in controlling the supersymmetry of the theory. In the specific example of Section 2.3, only negatively charged  $C_2$  fields appeared in the spectrum. The D-term potential generated by these fields exactly reproduced the regions of slope stability and instability shown in Figure 1. For a more general bundle, however, it is possible that *both* fields  $C_1, C_2$  in Table 2 are present in the spectrum. In this case, the  $U(1)$  D-term takes the form

$$D^{U(1)} = \frac{3}{16} \frac{\epsilon_S \epsilon_R^2}{\kappa_4^2} \frac{\mu(\mathcal{F})}{\mathcal{V}} - \frac{3}{2} \sum_{p, \bar{q}} G_{p\bar{q}} C_1^p \bar{C}_1^{\bar{q}} + \frac{3}{2} \sum_{P, \bar{Q}} G_{P\bar{Q}} C_2^P \bar{C}_2^{\bar{Q}} . \quad (2.21)$$

Now there are *two* terms available to cancel the Kähler moduli dependent FI-term. As we will see however, they play very different roles and  $C_1, C_2$  *can never obtain non-zero vevs simultaneously*.

To show this, first note, that in addition to the D-term (2.21), one must also consider the superpotential. Again ignoring  $E_6$  non-singlets, this can be written as <sup>8</sup>

$$W = \lambda_0 (C_1 C_2)^2 \quad (2.22)$$

where the indices on both fields and couplings are suppressed. In the stable region of Kähler moduli space, the four-dimensional effective theories we are considering have supersymmetric, Minkowski vacua. Therefore, as we vary the Kähler moduli away from the stability wall into the  $\mu(\mathcal{F}) < 0$  region of bundle  $V$ , we must preserve supersymmetry *and* avoid introducing a cosmological constant. The relevant equations, in addition to the vanishing of D-term (2.21), are

$$\begin{aligned} \partial_{C_1} W &= \lambda_0 C_2 (C_1 C_2) = 0 , \\ \partial_{C_2} W &= \lambda_0 C_1 (C_1 C_2) = 0 , \\ W &= \lambda_0 C_1 C_2 C_1 C_2 = 0 . \end{aligned} \quad (2.23)$$

With  $\mu(\mathcal{F}) < 0$  in (2.21), one might suppose that to preserve supersymmetry, the fields  $C_1$  and  $C_2$  could *both* get vevs such that the last two terms in  $D^{U(1)}$  cancel the FI term. However, substituting these two non-zero vevs into equations (2.23), it is clear that no such solution is possible. This is most easily verified by noting that, without loss of generality, one can choose a basis of field space so that only one of the  $C_1$  fields and one of the  $C_2$  fields has a non-vanishing vev. Thus, to move into the stable region of  $V$  and obtain a Minkowski vacuum, the only choice available is to take all  $\langle C_1^p \rangle = 0$  and to choose non-vanishing  $C_2^P$  vevs so that the first and last terms in (2.21) cancel. What happens in the chamber of  $V$  where  $\mu(\mathcal{F}) > 0$ ? Here, it would *appear* from (2.21), (2.23)

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<sup>8</sup>Our argument will be unchanged if we include in  $W$  all higher powers of  $C_1 C_2$ ; that is, if  $W \sim \sum_n (C_1 C_2)^n$ . Hence, we consider only the lowest order term.

that supersymmetry could still be preserved by the reverse happening; that is,  $C_1$  fields getting non-vanishing vevs while all  $C_2$  vevs are zero. However, as we show in the remainder of this subsection, within the context of our chosen geometry, i.e. the bundle  $V$  defined by (2.6), *only  $C_2$  fields can have non-zero vevs*. Hence, in the  $\mu(\mathcal{F}) > 0$  chamber of the Kähler cone supersymmetry is spontaneously broken by the D-term.

The key to explaining this fact, and distinguishing the fields  $C_1$  and  $C_2$ , can be found in the associated algebraic geometry. Although they behave as charged matter fields on the stability wall, the  $C_1, C_2$  fields can also be viewed geometrically as the moduli which control the “mixing” of the components of  $\mathcal{F} \oplus \mathcal{K}$  together to form an indecomposable bundle. To see this, recall how matter fields arise in a heterotic compactification. For dimensional reduction, the ten-dimensional  $E_8$  gauge fields,  $\mathcal{A}$ , on the “visible sector” fixed plane are expanded in a decomposition which is related to the bundle structure group. On the stability wall, the relevant ansatz is

$$\mathcal{A}_b = A_b + C_1^p \omega_{pb}^{(1)x} T_x^{(1)} + C_2^P \omega_{Pb}^{(2)y} T_y^{(2)} + \dots \quad (2.24)$$

Fields  $A_b$  are the gauge connection valued in  $S[U(2) \times U(1)]$ . The dots indicate terms involving other fields, such as  $F$ , from Table 2. From (2.15) we see that the adjoint of  $E_8$  breaks up into a series of pieces, one which is  $(\mathbf{1}, \mathbf{2})_3$  and another  $(\mathbf{1}, \mathbf{2})_{-3}$ , under the branching to  $E_6 \times SU(2) \times U(1)$ .  $T^{(1)}$  and  $T^{(2)}$  in (2.24) are precisely these gauge group generators, with the indices  $x$  and  $y$  running over the  $\mathbf{2}$  representation of  $SU(2)$ . The symbols  $\omega^{(1)}$  and  $\omega^{(2)}$  denote harmonic one-forms valued in  $\mathcal{F}^* \otimes \mathcal{K}$  and  $\mathcal{F} \otimes \mathcal{K}^*$  respectively. Hence, the number of  $C_2$  fields is found by counting the independent one-forms valued in  $\mathcal{F} \otimes \mathcal{K}^*$ , while the  $C_1$  fields arise as the independent one-forms valued in  $\mathcal{F}^* \otimes \mathcal{K}$ . This can be re-expressed in terms of Ext-groups [28] as

$$\begin{aligned} \text{Number of } C_2\text{'s} &= h^1(X, \mathcal{F} \otimes \mathcal{K}^*) = \dim(\text{Ext}^1(\mathcal{K}, \mathcal{F})) , \\ \text{Number of } C_1\text{'s} &= h^1(X, \mathcal{F}^* \otimes \mathcal{K}) = \dim(\text{Ext}^1(\mathcal{F}, \mathcal{K})) . \end{aligned} \quad (2.25)$$

From (2.24) we see that, when we give a  $C$  field a vev, the ten-dimensional gauge connection changes its expectation value. Equation (2.25) tells us what this change means in terms of bundle structure. The Ext-groups correspond to the moduli spaces of *two different extension bundles* [28, 14],

$$0 \rightarrow \mathcal{F} \rightarrow V \rightarrow \mathcal{K} \rightarrow 0 \quad \leftrightarrow \quad \text{Ext}^1(\mathcal{K}, \mathcal{F}) , \quad (2.26)$$

$$0 \rightarrow \mathcal{K} \rightarrow \tilde{V} \rightarrow \mathcal{F} \rightarrow 0 \quad \leftrightarrow \quad \text{Ext}^1(\mathcal{F}, \mathcal{K}) \quad (2.27)$$

respectively.  $V$  and  $\tilde{V}$  are referred to as an extension and its “dual” extension. They are *not in general isomorphic*.

It follows from (2.25), (2.26) that when  $\langle C_2 \rangle \neq 0$ ,  $A$  in (2.24) becomes an *irreducible* connection on  $V$ . Similarly, comparing (2.24), (2.25) and (2.27), we see that giving  $C_1$  a vev corresponds to  $A$  becoming an irreducible connection on  $\tilde{V}$ . However, since  $V$  and  $\tilde{V}$  are not isomorphic, for a given geometry, one can have *either* non-vanishing  $C_2$  *or* non-vanishing  $C_1$ , but *not both*. This is

the higher-dimensional manifestation of the statement derived in effective field theory earlier in this subsection:  $\langle C_1 \rangle$  and  $\langle C_2 \rangle$  can never be non-zero simultaneously. Note that the bundle  $V$  discussed at the beginning of this subsection is of the type (2.26). This explains why only its  $C_2$  fields can get a non-zero vev. Importantly, however, one could just as easily have analyzed the stability regions of  $\tilde{V}$  defined by (2.27), where  $C_1$  can be non-zero. These two “branches” of the vacuum space, where  $\langle C_2 \rangle \geq 0$ , and  $\langle C_1 \rangle \geq 0$ , respectively, intersect at exactly one locus, the stability wall, where both vevs vanish and the connection in  $A$  lives on the bundle  $\mathcal{F} \oplus \mathcal{K}$ . Thus, by changing the vevs of the four-dimensional fields, one can move smoothly between non-isomorphic internal gauge bundles for heterotic compactifications<sup>9</sup>. In the following, we will discuss the theory *corresponding to only one branch at a time*. A more detailed study of this stability wall induced branch structure, and transitions between such theories, will appear separately [60].

### 3 Wall Induced Yukawa Textures

We can now turn to the main question of this paper - can the existence of a stability wall constrain the physics of a compactification, even when the vacuum is in the interior of the stable region? The answer, as we will see, is affirmative. In this section, we continue to illustrate the main ideas using rank three bundles whose Kähler cone contains a stability wall where the  $SU(3)$  structure group decomposes into  $S[U(2) \times U(1)]$ . Thus, on and near this wall, the  $E_6$  gauge group is enhanced by a single  $U(1)$  factor, giving rise to one Abelian D-term in the effective theory. The types of conclusion drawn from this analysis remain unchanged for bundles of higher rank, and for stability walls with more than one D-term.

#### 3.1 Textures Near a Stability Wall

Consider a heterotic compactification associated with a bundle  $V$  of the form (2.6). *On and near the stability wall*, the superpotential is constrained by the gauge symmetry of the four-dimensional theory, *including the extra  $U(1)$* . Using Table 2, the relevant matter field superpotential consistent

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<sup>9</sup>Note that the D-term in (2.21), and its associated quantities, are only defined up to an overall sign. However, the relative sign between the  $C$  terms and the FI term in (2.21) is fixed, and arises from the choice of embedding of  $S[U(2) \times U(1)]$  (associated with  $\mathcal{F} \oplus \mathcal{K}$ ) inside  $SU(3)$  (associated with  $V$ ) [16] and anomaly cancellation [55]. Since we began by describing the geometry of the bundle in (2.6) and (2.26), here we have chosen the sign conventions in (2.21) so that the FI term is equal to a positive multiple of  $\mu(\mathcal{F})$ . Had we begun with (2.27) instead, the opposite sign convention could, of course, be taken. Note that since  $\mu(\mathcal{F}) = -\mu(\mathcal{K})$ , whichever sign convention is chosen, the sign of the FI term will be opposite in the two branches.



with gauge invariance is given by <sup>10</sup>

$$W = \lambda_0(C_1 C_2)^2 + \lambda_1 F_1^3 C_1^2 + \bar{\lambda}_1 \bar{F}_1^3 C_2^2 + \lambda_2 F_1^2 F_2 C_1 + \bar{\lambda}_2 \bar{F}_1^2 \bar{F}_2 C_2 \\ + \lambda_3 F_1 F_2^2 + \bar{\lambda}_3 \bar{F}_1 \bar{F}_2^2 + \lambda_4 F_2^3 C_2 + \bar{\lambda}_4 \bar{F}_2^3 C_1 . \quad (3.1)$$

Note that no quadratic terms appear, since all of these superfields are zero-modes of the compactification. Furthermore, terms of dimension six or higher in  $E_6$  non-singlet fields are not of interest to us, so we ignore them. Finally, we have displayed only the lowest dimensional terms required in our analysis. Each term can be multiplied by any positive integer power of  $C_1 C_2$ . Such terms do not change the subsequent analysis and, hence, in the interests of brevity, we suppress them.

*On the stability wall*  $\mu(\mathcal{F}) = 0$  and, hence, the FI term in (2.21) vanishes. In order to have both  $D^{U(1)} = 0$  and a solution to (2.23), it follows that  $\langle C_1 \rangle = \langle C_2 \rangle = 0$ . Substituting this into (3.1), the most general tri-linear couplings possible between  $E_6$  families on the stability wall are

$$W_{\text{Yukawa}}^{\text{wall}} = \lambda_3 F_1 F_2^2 + \bar{\lambda}_3 \bar{F}_1 \bar{F}_2^2 . \quad (3.2)$$

Note that only one type of coupling appears. All others, such as  $F_1^3$ , vanish. This is an *extremely* restrictive texture of Yukawa couplings. The fact that a Yukawa texture emerges precisely on the stability wall is, perhaps, of limited interest. Although some model building has been carried out on such a locus [27], it is more common to build standard model-like physics in the interior stable region. Let us analyze, therefore, what happens to the texture, (3.2), as we move into this chamber.

Consider a point in the stable region *close to, but not on*, the stability wall. Here  $\mu(\mathcal{F}) < 0$ , which implies, through the vanishing of the D-term (2.21) and equations (2.23), that  $\langle C_1 \rangle = 0$  and  $\langle C_2 \rangle \neq 0$ . Using this in (3.1), the allowed cubic matter couplings become

$$W_{\text{Yukawa}}^{\text{near wall}} = \lambda_3 F_1 F_2^2 + \bar{\lambda}_3 \bar{F}_1 \bar{F}_2^2 + \bar{\lambda}_1 \langle C_2^2 \rangle \bar{\mathbf{F}}_1^3 + \bar{\lambda}_2 \langle C_2 \rangle \bar{\mathbf{F}}_1^2 \bar{\mathbf{F}}_2 + \lambda_4 \langle C_2 \rangle \mathbf{F}_2^3 . \quad (3.3)$$

Note that the non-zero  $C_2$  vevs have allowed some Yukawa couplings missing in (3.2) to “grow back” from higher dimensional terms. These are expressed in boldface. This is not true of all Yukawa couplings however. Specifically, the  $F_1^3$  and  $\bar{F}_2^3$  terms are still forbidden, despite the fact that the extended  $U(1)$  gauge symmetry is spontaneously broken. That is, there remains a non-trivial texture.

Thus, we have demonstrated the existence of non-trivial Yukawa texture induced by a stability wall, even for *small* deformations of the moduli into the stable region. However, can one extend the analysis of this subsection to moduli deep in the interior of the stable chamber? To answer this, let us recall the effective field theory descriptions associated with 1) being *on* the wall, 2) *near* the wall and 3) *far* from the wall in the stable region. *On the wall*,  $\langle C_1 \rangle = \langle C_2 \rangle = 0$ , both  $C_1, C_2$  are massless and the  $U(1)$  vector boson has a non-zero mass given in (2.18). Since this

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<sup>10</sup>Note that if some of the fields, such as the  $C_1$ ’s, do not appear in the low energy spectrum, that is, if the cohomology  $H^1(\mathcal{F}^* \otimes \mathcal{K})$  in Table 2 vanishes as in the example of Subsection 2.3, then the following discussion will lead to even more restrictive Yukawa textures.

mass is significantly smaller than the compactification scale, the extended  $U(1)$  should *not* be integrated out of the low energy theory. The superpotential is then restricted by the  $U(1)$  charges to expression (3.1) and the Yukawa couplings to (3.2). Moving away from the wall,  $\langle C_1 \rangle = 0$  and  $\langle C_2 \rangle \neq 0$ . However, the non-zero vevs of the  $C_2$  fields enlarge the mass of the  $U(1)$  gauge boson via expression (2.19), give an equivalent mass to a linear combination of  $\delta t^k, \delta C_2$  and mass to one combination of  $C_1$  fields. As long as the mass of the  $U(1)$  gauge boson remains controllably below the compactification scale, the  $U(1)$  should still *not* be integrated out of the theory and the superpotential continues to be given by (3.1) and the Yukawa couplings by (3.3). This defines what it means to be *near the wall*. What happens *far from the wall*? By definition, this occurs when  $\langle C_2 \rangle$  approaches a value such that the two terms in (2.19) become of equal size. It then follows from (2.19) that the  $U(1)$  gauge boson and the  $\delta C_2$  masses, as well as the  $C_1$  mass, become as large as the compactification scale and, hence, these field must be integrated out of the effective theory. There are two consequences of this. First, the linear combination of  $C_2$  fields with the non-zero vev is no longer in the spectrum and, hence, one can not write higher dimension terms proportional to powers of  $\langle C_2 \rangle$  as in (3.3). Second, the  $\mathbf{27}^3$  that do occur are no longer *necessarily* constrained by the  $U(1)$  quantum numbers. Hence, it would appear that the Yukawa textures found near the stability wall do not necessarily persist into the interior of the stable region. However, as we now show, the Yukawa textures *do persist*. To prove this, we use the notion of holomorphy.

## 3.2 Holomorphy of the Superpotential and General Textures

For a generic heterotic compactification which preserves  $\mathcal{N} = 1$  supersymmetry, has an  $E_6$  GUT factor in its four dimensional gauge group, and has vanishing cosmological constant, any matter superpotential Yukawa coupling in the effective low-energy theory is of the form  $\lambda F^3$ , where  $F$  is either a  $\mathbf{27}$  or a  $\overline{\mathbf{27}}$  of  $E_6$ . Furthermore, each coefficient

$$\lambda = \lambda(\langle C_i \rangle, \langle \mathfrak{z}^a \rangle, \langle \phi \rangle) \quad (3.4)$$

must be a holomorphic function on the complex vacuum manifold  $\mathcal{M}$  of flat directions of the effective potential energy. Note that these couplings only depend upon the  $C$  and  $\phi$  fields, which we have already encountered, and the complex structure moduli of the Calabi-Yau threefold,  $\mathfrak{z}^a$ . Now consider the following general theorem.

- If a multivariate holomorphic function with domain  $U \subset \mathbb{C}^n$  vanishes on an open subset  $B \subset U$ , then it vanishes everywhere on  $U$  [58].

Let us identify  $U$  with a patch in an open cover of  $\mathcal{M}$  such that it contains  $B$ , an open subset, which covers a region on and near to the stability wall. We know from the preceding discussion that the coupling parameters of  $F_1^3$  and  $\bar{F}_2^3$  are both holomorphic functions on  $\mathcal{M}$  which do indeed vanish on such an open patch near the wall. By covering the vacuum space  $\mathcal{M}$  with open patches, overlapping on open intersections, we see from the above theorem that both the  $F_1^3$  and  $\bar{F}_2^3$  couplings must

vanish everywhere - that is, they vanish identically in the complete vacuum space, not just near the stability wall. An open cover of this form can be found on any smooth manifold. On the other hand, any Yukawa couplings, such as  $F_1 F_2^2$  or  $F_2^3$ , whose holomorphic parameters do *not* vanish in an open region near the wall, will not vanish anywhere in the interior of the stable chamber with the possible exception of isolated regions of higher co-dimension. We conclude that: *Yukawa textures appearing near the stability wall due to invariance under the extended  $U(1)$  charge, persist throughout the entire stable region, arbitrarily far from the wall, even though the  $U(1)$  has been integrated out of the theory. This result follows simply from the holomorphicity of the superpotential.*

We have been considering the branch of the vacuum where, near the wall,  $\langle C_2 \rangle \neq 0$  and  $\langle C_1 \rangle = 0$ . In general, as discussed in Subsection 2.6, there is second branch defined by (2.27), where  $\mu(\mathcal{K}) < 0$  (i.e.  $\mu(\mathcal{F}) > 0$ ),  $\langle C_1 \rangle \neq 0$  and  $\langle C_2 \rangle = 0$ . In this second branch, we see from (3.1) that there can be non-vanishing Yukawa couplings, such as  $\lambda_1 \langle C_1^2 \rangle F_1^3$  for example, that are absent in (3.3). How is this compatible with the above claim that if the Yukawa couplings vanish in an open subset of the vacuum space then they vanish everywhere? The answer is simply that, while each branch of the supersymmetric Minkowski vacuum space is a smooth manifold, the locus where they intersect is not. Such an intersection can not be covered with open sets with open intersections. In particular, denoting the two branches above by  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , if we take an open set  $B_1 \subset \mathcal{M}_1$  and another  $B_2 \subset \mathcal{M}_2$  then, if they intersect at all, their intersection must obey the condition  $\langle C_1 \rangle = \langle C_2 \rangle = 0$ . That is, their intersection is necessarily closed. One can, therefore, have a holomorphic function, such as the  $F_1^3$  Yukawa coupling, that is vanishing everywhere on one branch of the vacuum space and non-zero on the other - there being no overlapping open sets to “communicate” between the two. It follows that we have to make our previous conclusion more specific. That is: *In a given branch of the theory, Yukawa textures near a stability wall persist in the entire stable chamber of that branch. A stable region associated with a different vector bundle separated by a stability wall, that is, in a different branch of the theory, need not have identical Yukawa textures.*

### 3.3 A Higher-Dimensional Perspective

Before proceeding, let us analyze from a higher-dimensional perspective what is happening when Yukawa couplings “grow back”. Begin on the stability wall. Expand the dimensional reduction ansatz (2.24) to include, for example, the  $F$ -fields in Table 2. Then

$$\mathcal{A}_b = A_b + C_1^p \omega_{pb}^{(1)x} T_x^{(1)} + C_2^Q \omega_b^{(2)y} T_y^{(2)} + F_1^i \omega_{ib}^{(3)} T^{(3)} + F_2^\eta \omega_{\eta b}^{(4)z} T_z^{(4)} + \dots \quad (3.5)$$

The one-forms  $\omega$  are all harmonic with respect to the connection, built out of the background gauge field  $A$ , appropriate to the representation of the gauge group within which it is valued. Dimensional reduction then determines the Yukawa coupling parameters as integrals of the cubic product of these forms over the Calabi-Yau threefold. For example, the  $F_2^3$  Yukawa coupling is proportional too

$$\int_X f_{xyz} \omega^{(4)x} \wedge \omega^{(4)y} \wedge \omega^{(4)z} \wedge \overline{\Omega} \,, \quad (3.6)$$

where  $\Omega$  is the holomorphic three-form and  $f_{xyz}$  projects the wedge product of three one forms onto the gauge singlet. Thus, the texture (3.2) we observed on the stability wall arising from the extended  $U(1)$  gauge invariance can be viewed simply as the vanishing or non-vanishing of such integrals. For example, on the wall integral (3.6) vanishes, simply as a consequence of the contraction of the one forms with  $f$ .

As one moves away from the stability wall, the  $C$  fields must acquire non-zero vevs to cancel the FI term. *Near the stability wall*, where these vevs are small, their contribution to the connection can be dealt with perturbatively. Expanding  $C_i = \langle C_i \rangle + \delta C_i$ , instead of (3.5) one could write

$$\mathcal{A}_b = \hat{A}_b + \delta C_1^p \omega_{p\ b}^{(1)x} T_x^{(1)} + \delta C_2^Q \omega_b^{(2)y} T_y^{(2)} + F_1^i \hat{\omega}_{ib}^{(3)} T^{(3)} + F_2^\eta \hat{\omega}_{\eta b}^{(4)z} T_z^{(4)} + \dots, \quad (3.7)$$

where

$$\hat{A}_b = A_b + \langle C_1^p \rangle \omega_{p\ b}^{(1)x} T_x^{(1)} + \langle C_2^Q \rangle \omega_b^{(2)y} T_y^{(2)}. \quad (3.8)$$

The one-forms  $\hat{\omega}$  can now be taken to be harmonic with respect to connections built out of  $\hat{A}$  and the Yukawa coupling parameters become integrals of cubic products of *these* forms. For example, the  $F_2^3$  Yukawa coupling is now

$$\int_X f_{xyz} \hat{\omega}^{(4)x} \wedge \hat{\omega}^{(4)y} \wedge \hat{\omega}^{(4)z} \wedge \overline{\Omega}. \quad (3.9)$$

Thus, the texture (3.3) near the stability wall arising from the *spontaneous breaking* of extended  $U(1)$  gauge invariance can be viewed as the vanishing or non-vanishing of these integrals. As an example, (3.9) *no longer vanishes*.

What happens deep in the interior of a stable chamber? *Far from the stability wall*, the vevs of the  $C$  fields become so large that their contribution to the connection is comparable to  $A$ . At this stage, perturbative expansion (3.8) breaks down and (3.5) becomes

$$\mathcal{A}_b = \tilde{A}_b + F_1^i \tilde{\omega}_{ib}^{(3)} T^{(3)} + F_2^\eta \tilde{\omega}_{\eta b}^{(4)} T^{(4)} + \dots, \quad (3.10)$$

where the one-forms  $\tilde{\omega}$  are harmonic with respect to connections built out of  $\tilde{A}$ . Unfortunately, the indecomposable connection  $\tilde{A}$  is no longer related to the reducible connection  $A$  on the stability wall via a perturbative expansion. Hence, *a priori* one has no idea what the texture of the cubic  $\tilde{\omega}$  integrals are. Unlike the case near the wall, texture here cannot be found by inserting the non-zero  $C$  vevs into (3.1). However, the analysis of the preceding subsection shows that a connection between the two theories can indeed be determined using the holomorphicity of the superpotential.

With these observations on holomorphy and general textures in hand, we turn next to a more complicated, and restrictive, example of wall-induced Yukawa textures.

## 4 One Wall with Two D-Terms

In the previous section, we considered the case of a *single* stability wall in the Kähler cone where the bundle splits into *two* pieces. There are two immediate generalizations of this. Here, we

describe what happens in cases where the bundle splits into *more than two* pieces on a *single* wall. In the next section, we discuss the situation where multiple stability walls are present inside the Kähler cone. The simplest case where a bundle can split into more than two pieces occurs for an  $SU(3)$  structure group. Therefore, as previously, we will illustrate the main ideas using rank three bundles whose Kähler cone contains a single stability wall. Now, however, the structure group  $SU(3)$  decomposes into  $S[U(1) \times U(1) \times U(1)]$  on this wall. The conclusions drawn from this analysis remain unchanged for bundles of any rank.

Consider an  $SU(3)$  bundle  $V$  that splits on the stability wall, not as  $V = \mathcal{F} \oplus \mathcal{K}$  as in the previous section, but rather as

$$V = l_1 \oplus l_2 \oplus l_3 \quad (4.1)$$

where  $l_1, l_2$ , and  $l_3$  are line bundles. In the  $\mathcal{F} \oplus \mathcal{K}$  case,  $\mathcal{F}$  was the destabilizing sub-bundle, the vanishing of whose slope defined the stability wall. For the decomposition (4.1), let us take a case where two of the three line bundles destabilize  $V$ . Choosing such a pair of line bundles, the stability wall is defined by the simultaneous vanishing of their slopes. On this locus, the structure group changes from  $SU(3)$  to  $S[U(1) \times U(1) \times U(1)] \cong U(1) \times U(1)$ , where the last relation holds at least locally. Therefore, the low energy gauge group changes from  $E_6$  in the interior of any stable region to  $E_6 \times U(1) \times U(1)$  on the stability wall, and we get *two* extended Abelian gauge group factors rather than one. The analysis of Yukawa textures is similar to Section 3. Now, however, there are *two* D-terms of the form (2.21), one for each  $U(1)$  factor, and the theory near the stability wall is restricted by *both* extended Abelian symmetries. Despite this, the analytic continuation arguments of Section 3.2 remain unchanged. Hence, within a given branch, Yukawa textures near the stability wall persist over that entire stable chamber of the Kähler cone, arbitrarily far from the wall.

Given (4.1), the group theory which determines which multiplets can appear in the four-dimensional theory near the stability wall is

$$\begin{aligned} E_8 &\supset E_6 \times U(1) \times U(1) \\ \mathbf{248} &= \mathbf{1}_{0,0} + \mathbf{1}_{\frac{1}{2},3} + \mathbf{1}_{-\frac{1}{2},3} + \mathbf{1}_{\frac{1}{2},-3} + \mathbf{1}_{-\frac{1}{2},-3} + \mathbf{1}_{1,0} + \mathbf{1}_{-1,0} + \mathbf{78}_{0,0} \\ &\quad + \mathbf{27}_{0,-2} + \mathbf{27}_{\frac{1}{2},1} + \mathbf{27}_{-\frac{1}{2},1} + \overline{\mathbf{27}}_{0,2} + \overline{\mathbf{27}}_{\frac{1}{2},-1} + \overline{\mathbf{27}}_{-\frac{1}{2},-1} , \end{aligned} \quad (4.2)$$

where the bold face number is the dimension of the  $E_6$  representation and the subscripts are the two  $U(1)$  charges. The multiplicity of each such multiplet is determined by the number of zero-modes of the associated six-dimensional Dirac operator. These are given by the dimensions of bundle-valued cohomology groups. The representations, field names and the associated cohomologies for a *generic* bundle of type (4.1) are listed in the first three columns of Table 3. Note that there are now *six* different types of charged bundle moduli, that is,  $C$ -fields, of the kind described in Section 2.6. As discussed in that subsection, the  $C$  fields are intimately related to the branch structure of the theory. We now generalize the analysis of Section 2.6 to the present case.

First choose which two line bundles in (4.1) destabilize  $V$ . Let us take  $l_1$  and  $l_2$  for specificity. Associated with each will be a D-term of the generic form (2.21), where  $D_1^{U(1)}$  and  $D_2^{U(1)}$  contain

the slope  $\mu(l_1)$  and  $\mu(l_2)$  respectively. Although both slopes vanish on the stability wall, the assumption that the associated line bundles destabilize  $V$  implies that their slopes become negative in the interior of the stable chamber. Now note that, in addition to these two D-terms, one must consider the superpotential. Ignoring the  $E_6$  non-singlets, this can be written as

$$W = C_1\tilde{C}_2\tilde{C}_3 + (C_1\tilde{C}_1)^2 + (C_2\tilde{C}_2)^2 + (C_3\tilde{C}_3)^2 + C_1C_2\tilde{C}_1\tilde{C}_2 + C_1C_3\tilde{C}_1\tilde{C}_3 + C_2C_3\tilde{C}_2\tilde{C}_3 . \quad (4.3)$$

For simplicity, here and in the remainder of the paper, we suppress indices and the coefficients in front of each term. In addition, we work only to the dimension required for our analysis. In any stable region, the four-dimensional effective theory has a supersymmetric vacuum with vanishing cosmological constant. Therefore, as we vary the Kähler moduli away from the stability wall into the  $\mu(l_1) < 0$ ,  $\mu(l_2) < 0$  region, in addition to the vanishing of the two D-terms, we must set

$$\partial_{C_i} W = \partial_{\tilde{C}_j} W = W = 0 \quad i, j = 1, 2, 3 . \quad (4.4)$$

Terms of the form  $(C_i\tilde{C}_i)^2$ ,  $i = 1, 2, 3$  in (4.3) ensure that, for each index  $i$ , either  $C_i$  or  $\tilde{C}_i$ , but not both, can have a non-zero vev. The terms of the form  $C_1\tilde{C}_2\tilde{C}_3$  and  $\tilde{C}_1C_2C_3$  in (4.3) ensure that only one of  $C_1$ ,  $\tilde{C}_2$  and  $\tilde{C}_3$ , and only one of  $\tilde{C}_1$ ,  $C_2$  and  $C_3$ , obtains a non-zero vev. Combining these results with the requirement that  $D_1^{U(1)} = D_2^{U(1)} = 0$ , we find that there are *six* supersymmetric branches associated with this stability wall in the Kähler cone - each branch specified by a pair of non-vanishing  $C$  fields. For example, one branch is given by

$$\langle \tilde{C}_2 \rangle \neq 0 , \quad \langle C_3 \rangle \neq 0 \quad (4.5)$$

where all other  $\langle C \rangle = 0$ .

In terms of sequences, the different possible  $C$  field vevs correspond to the different ways of building a bundle  $V$  from the three constituent line bundles  $l_1$ ,  $l_2$  and  $l_3$ . Let us take the case (4.5), where  $\tilde{C}_2$  and  $C_3$  have non-zero vevs, as a specific example. Consider the two sequences

$$0 \rightarrow l_1 \rightarrow \mathcal{W} \rightarrow l_3 \rightarrow 0 \quad (4.6)$$

$$0 \rightarrow l_2 \rightarrow V \rightarrow \mathcal{W} \rightarrow 0 \quad (4.7)$$

The moduli space of the first sequence is described by  $Ext^1(l_3, l_1) \cong H^1(l_1 \otimes l_3^*)$ . Therefore, this extension is non-trivial, that is,  $\mathcal{W} \neq l_1 \oplus l_3$ , if and only if one is at a non-zero element of this cohomology group. We see from Table 3 that this corresponds, in field theory language, to  $\langle \tilde{C}_2 \rangle \neq 0$ . Sequence (4.7) is a non-trivial extension if and only if one is at a non-trivial element in  $Ext^1(\mathcal{W}, l_2) \cong H^1(l_2 \otimes \mathcal{W}^*)$ . Using the dual sequence to (4.6), we find

$$0 \rightarrow l_2 \otimes l_3^* \rightarrow l_2 \otimes \mathcal{W}^* \rightarrow l_2 \otimes l_1^* \rightarrow 0 . \quad (4.8)$$

Since all vevs for the  $\tilde{C}_1$  fields vanish, it follows from Table 3 that this branch is confined to the zero-element of  $H^1(l_2 \otimes l_1^*)$ . The long exact sequence associated with (4.8) then simplifies to

$$\dots \rightarrow H^1(l_2 \otimes l_3^*) \rightarrow H^1(l_2 \otimes \mathcal{W}^*) \rightarrow 0 . \quad (4.9)$$

It follows that any non-zero element of  $H^1(l_2 \otimes l_3^*)$ , the cohomology associated with the fields  $C_3$ , maps to a non-zero element of  $H^1(l_2 \otimes \mathcal{W}^*)$ , the cohomology associated with bundle (4.7). That is, the deviation of the bundle,  $V$ , away from its split point in sequence (4.7) is controlled by the  $\langle C_3 \rangle \neq 0$  condition in the field theory. Putting everything together, we conclude that  $V$  in (4.7) is indeed the bundle corresponding to the branch of the vacuum space where  $\langle \tilde{C}_2 \rangle \neq 0, \langle C_3 \rangle \neq 0$  and all other  $C$  vevs vanish. A similar analysis can be performed for any other allowed branch.

We now turn to an analysis of the allowed Yukawa textures. All of the fields in Table 3, if present in a specific example, are massless near the wall. The most general superpotential for

Representation	Field Name	Cohomology	Multiplicity
$\mathbf{1}_{\frac{1}{2},3}$	$C_1$	$h^1(X, l_1^* \otimes l_2)$	0
$\mathbf{1}_{-\frac{1}{2},-3}$	$\tilde{C}_1$	$h^1(X, l_1 \otimes l_2^*)$	0
$\mathbf{1}_{-\frac{1}{2},3}$	$C_2$	$h^1(X, l_1^* \otimes l_3)$	0
$\mathbf{1}_{\frac{1}{2},-3}$	$\tilde{C}_2$	$h^1(X, l_1 \otimes l_3^*)$	100
$\mathbf{1}_{1,0}$	$C_3$	$h^1(X, l_2 \otimes l_3^*)$	200
$\mathbf{1}_{-1,0}$	$\tilde{C}_3$	$h^1(X, l_2^* \otimes l_3)$	0
$\mathbf{27}_{0,-2}$	$f_1$	$h^1(X, l_1)$	0
$\mathbf{27}_{\frac{1}{2},1}$	$f_2$	$h^1(X, l_2)$	20
$\mathbf{27}_{-\frac{1}{2},1}$	$f_3$	$h^1(X, l_3)$	0
$\mathbf{27}_{0,2}$	$\tilde{f}_1$	$h^1(X, l_1^*)$	0
$\mathbf{27}_{-\frac{1}{2},-1}$	$\tilde{f}_2$	$h^1(X, l_2^*)$	0
$\mathbf{27}_{\frac{1}{2},-1}$	$\tilde{f}_3$	$h^1(X, l_3^*)$	40

Table 3: The representations, field content and cohomologies of a *generic*  $E_6 \times U(1) \times U(1)$  theory associated with a poly-stable bundle  $V = l_1 \oplus l_2 \oplus l_3$  on the stability wall. The multiplicities for the *explicit* bundle defined by (4.16) are given in the fourth column.

cubic matter interactions invariant under the  $E_6 \times U(1) \times U(1)$  symmetry, including the purely  $C$  field superpotential in (4.3), is given by

$$\begin{aligned}
W = & f_1 f_2 f_3 + C_1 \tilde{C}_2 \tilde{C}_3 \\
& + (C_1 \tilde{C}_1)^2 + (C_2 \tilde{C}_2)^2 + (C_3 \tilde{C}_3)^2 + C_1 C_2 \tilde{C}_1 \tilde{C}_2 + C_1 C_3 \tilde{C}_1 \tilde{C}_3 + C_2 C_3 \tilde{C}_2 \tilde{C}_3 \\
& + f_1^2 f_2 C_2 + f_1^2 f_3 C_1 + f_2^2 f_1 \tilde{C}_3 + f_2^2 f_3 \tilde{C}_1 + f_3^2 f_1 C_3 + f_3^2 f_2 \tilde{C}_2 \\
& + f_1^3 C_1 C_2 + f_1^2 f_2 C_1 \tilde{C}_3 + f_1^2 f_3 C_2 C_3 + f_1 f_2^2 \tilde{C}_1 C_2 + f_1 f_3^2 C_1 \tilde{C}_2 \\
& + f_2^3 \tilde{C}_1 \tilde{C}_3 + f_2^2 f_3 \tilde{C}_2 \tilde{C}_3 + f_2 f_3^2 \tilde{C}_1 C_3 + f_3^3 \tilde{C}_2 C_3 \\
& + f_1^3 C_1^2 \tilde{C}_3 + f_1^3 C_2^2 C_3 + f_2^3 \tilde{C}_1^2 C_2 + f_2^3 \tilde{C}_2 \tilde{C}_3^2 + f_3^3 C_1 \tilde{C}_2^2 + f_3^3 \tilde{C}_1 C_3^2 \\
& + \{f_1, f_2, f_3, C_1, C_2, C_3\} \leftrightarrow \{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3\} ,
\end{aligned} \tag{4.10}$$

where terms are shown in the order of increasing dimension and we do not display any coefficients or indices. No quadratic terms appear, since all these superfields are zero-modes of the compactification. Furthermore, interactions of dimension six or higher in  $E_6$  non-singlet fields  $f$  and  $\tilde{f}$  are not relevant to the discussion, so we ignore them. Finally, displayed are the lowest dimension terms required in our analysis. Each interaction in (4.10) can be multiplied by any positive integer power of neutral combinations of  $C$  fields. These do not change the subsequent analysis and, hence, in the interests of brevity, we suppress them.

Examining (4.10), we see that the only Yukawa couplings present *on the stability wall*, where all  $C$  field vevs vanish, are

$$W_{\text{Yukawa}}^{\text{wall}} = f_1 f_2 f_3 + \tilde{f}_1 \tilde{f}_2 \tilde{f}_3 . \quad (4.11)$$

This is a very restrictive texture. As in the previous section, some of the missing Yukawa couplings can “grow back” as one moves away from the stability wall into a stable chamber of the Kähler cone. Returning to (4.10), it is clear that there are several possible Yukawa textures that can result from the the splitting of an  $SU(3)$  bundle into three line bundles on the stability wall. Which texture occurs depends on which  $C$  fields get non-zero vevs, that is, which branch of the theory one is on. For the representative branch discussed above, where  $\langle \tilde{C}_2 \rangle \neq 0, \langle C_3 \rangle \neq 0$  and all other  $C$  vevs vanish, we find that

$$\begin{aligned} W_{\text{Yukawa}}^{\text{near wall}} = & f_1 f_2 f_3 + \tilde{f}_1 \tilde{f}_2 \tilde{f}_3 + \langle \mathbf{C}_3 \rangle \mathbf{f}_3^2 \mathbf{f}_1 + \langle \tilde{\mathbf{C}}_2 \rangle \mathbf{f}_3^2 \mathbf{f}_2 \\ & + \langle \tilde{\mathbf{C}}_2 \rangle \tilde{\mathbf{f}}_1^2 \tilde{\mathbf{f}}_2 + \langle \mathbf{C}_3 \rangle \tilde{\mathbf{f}}_2^2 \tilde{\mathbf{f}}_1 + \langle \tilde{\mathbf{C}}_2 \rangle \langle \mathbf{C}_3 \rangle \mathbf{f}_3^3 . \end{aligned} \quad (4.12)$$

Comparing to (4.11), it follows that there are five different types of Yukawa couplings which can “grow back” as we deform to the indecomposable bundle described by (4.6) and (4.7). These are shown in boldface. Be this as it may, it is important to note that there remain *many* Yukawa couplings, such as  $f_1^3, \tilde{f}_2^3, f_2 \tilde{f}_1^2$ , which are forbidden by the the extended  $U(1) \times U(1)$  symmetry. Finally, using the holomorphy analysis from subsection 3.2, we conclude that *everywhere* in this stable chamber the Yukawa texture is given by

$$W_{\text{Yukawa}} = f_1 f_2 f_3 + \tilde{f}_1 \tilde{f}_2 \tilde{f}_3 + \mathbf{f}_3^2 \mathbf{f}_1 + \mathbf{f}_3^2 \mathbf{f}_2 + \tilde{\mathbf{f}}_1^2 \tilde{\mathbf{f}}_2 + \tilde{\mathbf{f}}_2^2 \tilde{\mathbf{f}}_1 + \mathbf{f}_3^3 . \quad (4.13)$$

## An Example

As an example of a stability wall of the type discussed in this section, consider the bundle

$$0 \rightarrow \mathcal{O}(-1, 1) \oplus \mathcal{O}(-2, 2) \rightarrow V \rightarrow \mathcal{O}(3, -3) \rightarrow 0 \quad (4.14)$$

defined on the complete intersection Calabi-Yau threefold

$$\left[ \begin{array}{c|c} \mathbb{P}^1 & 2 \\ \mathbb{P}^3 & 4 \end{array} \right]^{2,86} . \quad (4.15)$$



Equation (4.14) describes  $V$  as an extension of direct sums of line bundles. This bundle has a stability wall on the line of slope one in its two-dimensional Kähler cone. On this locus, the bundle splits as

$$V = \mathcal{O}(-1, 1) \oplus \mathcal{O}(-2, 2) \oplus \mathcal{O}(3, -3) . \quad (4.16)$$

Hence, we can identify  $l_1$ ,  $l_2$  and  $l_3$  of the previous discussion as  $\mathcal{O}(-1, 1)$ ,  $\mathcal{O}(-2, 2)$  and  $\mathcal{O}(3, -3)$  respectively.

Given this explicit example, one can calculate the multiplicity of each multiplet described in (4.2). These are presented in the fourth column of Table 3. Note that there are 20  $f_2$  fields and no other **27** multiplets of  $E_6$ . Additionally, there are 40  $\tilde{f}_3$  fields but no other  $\overline{\mathbf{27}}$  anti-generations. Importantly, the only  $C$  fields which appear are  $\tilde{C}_2$  and  $C_3$ , precisely the fields that got non-zero vevs in our preceding discussion. Combining this information with (4.13), we find that there are *no Yukawa couplings at all*, either between three **27**'s or between three  $\overline{\mathbf{27}}$ 's. The residual symmetries left over in the interior of this stable region, as a result of the presence of the stability wall, completely remove all couplings between the matter families in this example. This is a good illustration of the importance of residual symmetries. In some cases they can be extremely restrictive, and could forbid some, or all, of the interactions required by phenomenology.

## 5 Two Walls in the Kähler Cone

In Section 3, we considered a single stability wall in the Kähler cone where the bundle split into two pieces. This was generalized in the preceding section to the case of a bundle which split into three or more pieces, again on a single stability wall. More generally, however, a supersymmetric chamber in the Kähler cone can be surrounded by multiple stability walls. In this section, we turn our attention to this situation. The simplest examples occur in vacua with  $h^{1,1} = 2$ . In this case, there can be at most two stability walls bounding a supersymmetric region (see Figure 2). In the most basic examples the vector bundle splits into just two pieces on each wall. We will restrict our discussion to rank three cases in order to illustrate these multi-wall scenarios, and the Yukawa textures they give rise to. We emphasize, however, that the general type of conclusions drawn from this analysis remain unchanged for  $h^{1,1} \geq 3$ , vector bundles of any rank, and for more general decompositions of the bundle.

Clearly, the analysis of Sections 2 and 3 applies to each of the two stability walls individually. Each wall, therefore, places constraints on the terms in the four-dimensional theory. The question of exactly how these constraints interrelate, however, is not easily answered. Note that the description of the effective theory, including the labeling of the fields, and possibly even the number of vector-like pairs, changes between boundaries. For example, consider a vacuum which, in the interior of the stable region, has three chiral **27** matter families of  $E_6$ . Furthermore, assume that at the “upper” stability wall no family/anti-family pairs appear and that two families get charge  $q_1$

and one family charge  $Q_1$  under the extended  $U(1)$  symmetry. Now suppose that at the “lower” boundary this second stability wall gives two families of charge  $q_2$  and one of charge  $Q_2$  under its extra Abelian gauge group. In general then, every field in the problem can carry two additional quantum numbers, one associated with the  $U(1)$  at the upper boundary and the second with the Abelian symmetry at the lower boundary. Each  $U(1)$  gauge symmetry only appears near the stability wall which gives rise to it, and so the associated charges only have meaning in that part of field space. We have two “near wall” theories with no overlapping region of validity. Given this, when we consider the fields at a generic point in the slope-stable region, how do we correlate the charges which they acquire as we near each of the two walls? For example, do we have two charged objects which pick up charge  $q_1$  near one wall and  $q_2$  near the other, or perhaps simply one and some fields with charges  $Q_1$  and  $q_2$ ? As may be expected, answering this type of question and, hence, describing the physics of multiple stability walls is, in general, example dependent. To untangle the most general constraints on the theory requires a careful observation of the chosen fields and geometry. There are some cases where the result is particularly simple, however, and we will give an illustrative example here.

To begin, consider the complete intersection manifold

$$X = \left[ \begin{array}{c|c} \mathbb{P}^1 & 2 \\ \hline \mathbb{P}^3 & 4 \end{array} \right]^{2,86}. \quad (5.1)$$

Over this space, we construct the following  $SU(3)$  monad bundle,

$$0 \rightarrow V \rightarrow \mathcal{O}(1,1) \oplus \mathcal{O}(2,0) \oplus \mathcal{O}(3,-1) \oplus \mathcal{O}(-2,1) \rightarrow \mathcal{O}(4,1) \rightarrow 0. \quad (5.2)$$

A stability analysis as in [16, 19, 23] reveals that this bundle is destabilized by a pair of rank two sub-bundles, namely

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{O}(1,1) \oplus \mathcal{O}(2,0) \oplus \mathcal{O}(-2,1) \rightarrow \mathcal{O}(4,1) \rightarrow 0 \quad (5.3)$$

where  $c_1(\mathcal{F}_1) = (-3, 1)$  and

$$0 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{O}(1,1) \oplus \mathcal{O}(2,0) \oplus \mathcal{O}(3,-1) \rightarrow \mathcal{O}(4,1) \rightarrow 0 \quad (5.4)$$

with  $c_1(\mathcal{F}_2) = (2, -1)$ . The vanishing of the slope of the first of these sub-bundles,  $\mathcal{F}_1$ , provides a “lower” boundary wall to the stable region of Kähler moduli space, while the vanishing of the slope of the second,  $\mathcal{F}_2$ , provides an “upper” boundary to this region. The stability wall structure for this example is given in Figure 2. We will consider each boundary in turn, and the effective field theory associated with it, before combining our observations to find the constraints on the full theory at a generic point in the stable region.

Let us begin our analysis on the “lower” boundary wall defined by the sub-bundle  $\mathcal{F}_1$ . For the theory to be supersymmetric, the bundle  $V$  in (5.2) must “split” on this stability wall into the direct sum  $\mathcal{F}_1 \oplus \mathcal{K}_1$ , where  $\mathcal{K}_1 = \mathcal{O}(3, -1)$ . The field content near this boundary, and the charges

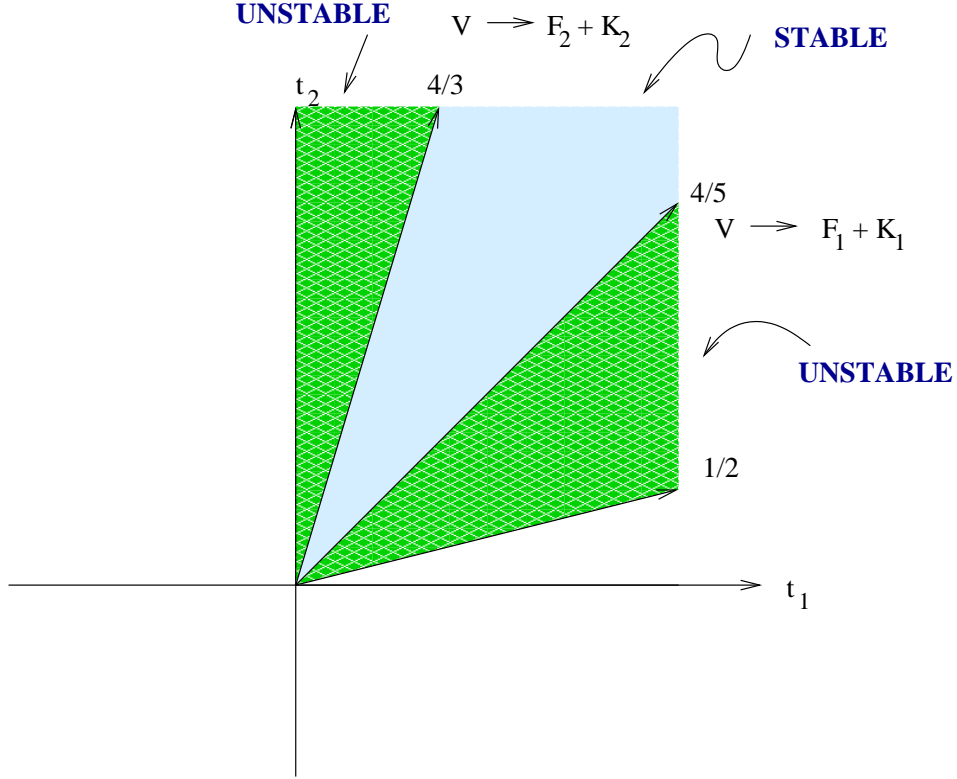


Figure 2: The Kähler cone (The set of moduli  $t^1, t^2 > 0$  and  $2t^2 > t^1$ ) and the regions of stability/instability for the Calabi-Yau threefold (5.1) and the bundle (5.2). At the lower boundary,  $V$  decomposes as  $V \rightarrow \mathcal{F}_1 \oplus \mathcal{K}_1$ , where  $\mathcal{F}_1$  is defined in (5.3). At the upper boundary, the poly-stable decomposition is given by  $V \rightarrow \mathcal{F}_2 \oplus \mathcal{K}_2$ , with  $\mathcal{F}_2$  defined by (5.4).

under the extended  $E_6 \times U(1)$  gauge symmetry, are given in Table 4. To third order in the matter fields, the invariant superpotential is

$$W = f_3^2 f_4 + f_3^3 C_1 . \quad (5.5)$$

As always, we ignore irrelevant higher dimension terms. Note that on the lower stability wall, all  $\mathbf{27}^3$  Yukawa couplings are forbidden entirely. Furthermore, the  $\overline{\mathbf{27}}^3$  couplings exhibit a very restrictive texture. For example, the  $f_i^3$ ,  $i = 1, 2, 3$  terms are absent. What happens for small deformations away from this wall into the stable chamber? Since one can describe the stable bundle  $V$  in terms of this de-stabilizing sub-bundle as

$$0 \rightarrow \mathcal{F}_1 \rightarrow V \rightarrow \mathcal{K}_1 \rightarrow 0 , \quad (5.6)$$

we see that  $C_1 \in H^1(X, \mathcal{F}_1 \times \mathcal{K}_1^*)$  must acquire a non-zero vev in order to cancel the FI piece of the D-term associated with the upper boundary, see (2.17). As a result, the  $\overline{\mathbf{27}}^3$  Yukawa coupling  $\langle C_1 \rangle f_3^3$  can “grow back” near this stability wall. It follows from the holomorphy analysis of subsection 3.2 that one expects

$$W_{\text{Yukawa}} = f_3^2 f_4 + \mathbf{f}_3^3 \quad (5.7)$$

Representation	Field Name	Cohomology	Multiplicity
$(\mathbf{27}, \mathbf{2})_{-1}$	$f_1$	$H^1(X, \mathcal{F}_1)$	13
$(\overline{\mathbf{27}}, \mathbf{2})_1$	$f_3$	$H^1(X, \mathcal{F}_1^*)$	1
$(\overline{\mathbf{27}}, \mathbf{1})_{-2}$	$f_4$	$H^1(X, \mathcal{K}_1^*)$	8
$(\mathbf{1}, \mathbf{2})_{-3}$	$C_1$	$H^1(X, \mathcal{F}_1 \otimes \mathcal{K}_1^*)$	70

Table 4: Field content on the lower stability wall of the *explicit* bundle (5.2). Cohomologies with vanishing multiplicity are not shown.

Representation	Field Name	Cohomology	Multiplicity
$(\mathbf{27}, \mathbf{2})_{-1}$	$\tilde{f}_1$	$H^1(X, \mathcal{F}_2)$	9
$(\overline{\mathbf{27}}, \mathbf{2})_1$	$\tilde{f}_3$	$H^1(X, \mathcal{F}_2^*)$	9
$(\mathbf{27}, \mathbf{1})_2$	$\tilde{f}_2$	$H^1(X, \mathcal{K}_2)$	4
$(\mathbf{1}, \mathbf{2})_{-3}$	$\tilde{C}_1$	$H^1(X, \mathcal{F}_2 \otimes \mathcal{K}_2^*)$	3
$(\mathbf{1}, \mathbf{2})_3$	$\tilde{C}_2$	$H^1(X, \mathcal{F}_2^* \otimes \mathcal{K}_2)$	49

Table 5: Field content on the upper stability wall of *explicit* bundle (5.2). Cohomologies with vanishing multiplicity are not shown.

*everywhere* in the interior of the stable chamber.

Turn now to the upper boundary, where the theory is defined by sub-bundle  $\mathcal{F}_2$  in (5.4). The polystable decomposition is now  $V \rightarrow \mathcal{F}_2 \oplus \mathcal{K}_2$ , with  $\mathcal{K}_2 = \mathcal{O}(-2, 1)$ . This is associated again with an extended  $E_6 \times U(1)$  symmetry of the four-dimensional theory. However, the  $U(1)$  symmetry near this wall is *not* the same as the Abelian symmetry on the lower boundary. Near the upper boundary, the field content and charges under the *new* extended  $E_6 \times U(1)$  gauge symmetry are listed in Table 5. The relevant superpotential is now given by

$$W = \tilde{f}_1^2 \tilde{f}_2 + \tilde{f}_2^2 \tilde{f}_1 \tilde{C}_1 + \tilde{f}_1^3 \tilde{C}_2 + \tilde{f}_3^3 \tilde{C}_1 + \tilde{f}_2^3 \tilde{C}_1^2 + \tilde{f}_1^2 \tilde{f}_2 \tilde{C}_1 \tilde{C}_2 + (\tilde{C}_1 \tilde{C}_2)^2 . \quad (5.8)$$

Note that *on* the upper stability wall, the  $\mathbf{27}^3$  Yukawa couplings exhibit a very restrictive texture. Furthermore, all  $\overline{\mathbf{27}}^3$  terms are disallowed. What happens for small deformations away from this wall into the stable chamber?

Note that while both charged moduli  $\tilde{C}_1, \tilde{C}_2$  are present (recall that these are the moduli responsible for “re-mixing”  $\mathcal{F}_2 \oplus \mathcal{K}_2$ , into  $V$  in (5.2)), as discussed in Section 2.6, only one of them can get a vev in the stable region. Since  $\mathcal{F}_2$  in  $V$  is the destabilizing sub-bundle at the upper boundary, it is clear that we can describe  $V$  in the stable region as

$$0 \rightarrow \mathcal{F}_2 \rightarrow V \rightarrow \mathcal{K}_2 \rightarrow 0 . \quad (5.9)$$

As a result, it is  $\tilde{C}_1 \in H^1(X, \mathcal{F}_2 \times \mathcal{K}_2^*)$  which controls the movement away from the upper stability wall into the indecomposable gauge configuration. This can also be seen by inspecting the charges of

the various fields in the  $U(1)$  D-term associated with the upper boundary, see (2.21). It follows from the slope of  $\mathcal{F}_2$  in Figure 2 that it is the  $\tilde{C}_1$  fields that must acquire a non-zero vev. Furthermore, the  $(\tilde{C}_1\tilde{C}_2)^2$  term in (5.8) assures that the vevs of  $\tilde{C}_2$  must be zero in the stable region. As a result, of the five  $C$  field dependent matter couplings in (5.8), three “grow back” to contribute to Yukawa couplings near this stability wall. It follows from the holomorphy analysis of subsection 3.2 that one expects

$$W_{\text{Yukawa}} = \tilde{f}_1^2 \tilde{f}_2 + \tilde{\mathbf{f}}_2^2 \tilde{\mathbf{f}}_1 + \tilde{\mathbf{f}}_3^3 + \tilde{\mathbf{f}}_2^3 \quad (5.10)$$

*everywhere* in the interior of the stable chamber.

Now consider the theory deep in the stable region, away from these two boundaries. Using (5.2), one can find the spectrum of the “standard” heterotic compactification in the stable region. Since  $V$  is a stable  $SU(3)$  bundle,  $H^0(X, V) = H^3(X, V) = 0$  [2, 17]. It follows that the long exact sequence in cohomology associated with (5.2) splits into

$$0 \rightarrow H^0(X, \mathcal{O}(1, 1) \oplus \mathcal{O}(2, 0)) \rightarrow H^0(X, \mathcal{O}(4, 1)) \rightarrow H^1(X, V) \rightarrow H^1(X, \mathcal{O}(-2, 1)) \rightarrow 0 \quad (5.11)$$

and

$$0 \rightarrow H^2(X, V) \rightarrow H^2(X, \mathcal{O}(3, -1) \oplus \mathcal{O}(2, 0)) \rightarrow 0. \quad (5.12)$$

As a result, we see that

$$h^1(X, V) = h^0(X, \mathcal{O}(4, 1)) - h^0(X, \mathcal{O}(1, 1) \oplus \mathcal{O}(2, 0)) + h^1(X, \mathcal{O}(-2, 1)) = 20 - 11 + 4 = 13 \quad (5.13)$$

and

$$h^1(X, V^*) = h^2(X, V) = h^2(X, \mathcal{O}(3, -1) \oplus \mathcal{O}(2, 0)) = 9 \quad (5.14)$$

respectively. Hence, in the interior of the stable chamber there are thirteen **27** and nine  $\overline{\mathbf{27}}$  multiplets of  $E_6$ ; that is, four chiral and nine vector-like pairs of matter families. How then does this general theory relate to the effective theories at each boundary wall? To answer this, consider the alternative descriptions of  $V$  in terms of  $\mathcal{F}_1$ , (5.6), and in terms of  $\mathcal{F}_2$ , (5.9). From the associated long exact cohomology sequences, we find that in the stable region

$$H^1(X, V) = H^1(X, \mathcal{F}_1) = H^1(X, \mathcal{F}_2) \oplus H^1(X, \mathcal{K}_2), \quad (5.15)$$

$$H^1(X, V^*) = H^1(X, \mathcal{F}_1^*) \oplus H^1(X, \mathcal{K}_1^*) = H^1(X, \mathcal{F}_2^*). \quad (5.16)$$

It then follows from Tables 4 and 5 that

$$H^1(X, V) = \text{span}\{f_1\} = \text{span}\{\tilde{f}_1, \tilde{f}_2\} \quad 13 = (9 + 4), \quad (5.17)$$

$$H^1(X, V^*) = \text{span}\{f_3, f_4\} = \text{span}\{\tilde{f}_3\} \quad (1 + 8) = 9. \quad (5.18)$$

The key point in this example is that, on the lower stability wall all of the families acquire the same charge. Equally, on the upper stability wall all of the anti-families acquire the same charge. Thus, we are able to correlate the charges picked up by the matter fields at the two different walls in an unambiguous manner! Note that the number of chiral families and the number of vector-like pairs

stays the same throughout moduli space. Using this result, one can now impose the constraints from *each* stability wall on couplings throughout the *entire* Kähler cone.

First observe from (5.7) and (5.17) that the constraints from the lower wall completely *forbid* any  $\mathbf{27}^3$  Yukawa couplings in the stable chamber. It then follows from (5.17) that the couplings  $\tilde{f}_1^2 \tilde{f}_2$ ,  $\tilde{\mathbf{f}}_2^2 \tilde{\mathbf{f}}_1$  and  $\tilde{\mathbf{f}}_2^3$  in (5.10), while not forbidden by gauge invariance at the upper boundary, are none-the-less vanishing everywhere due to the constraints from the lower wall. Second, observe from (5.7) and (5.18) that the lower wall constraints do allow  $\overline{\mathbf{27}}^3$  Yukawa couplings in the stable chamber, but only in a *specific texture with nine terms*. We see from (5.18) that this is group theoretically consistent with the existence of the  $\tilde{\mathbf{f}}_3^3$  in (5.10). However, the gauge symmetry of the upper wall would allow  $\binom{9}{3} + 9(9-1) + 9 = 165$  such terms. It follows that additional texture is imposed by the constraints of the lower wall to reduce this number to 9. Note from (5.7) and (5.10) that these nine holomorphic parameters, while non-vanishing in the interior of the stable region, must depend on bundle moduli in such a way that they all go to zero at the upper boundary, while eight remain non-zero at the lower wall.

We conclude that it is possible to trace the constraints from both boundary stability walls into the interior of the stable chamber. At a generic point of this four-generation, nine vector-like pair  $E_6$  theory, we find that there are no  $\mathbf{27}^3$  couplings allowed and only 9 specific  $\overline{\mathbf{27}}^3$  Yukawa couplings surviving out of 165. Note that the stronger constraints arise from the bottom stability wall. Had one only considered constraints from the upper stability wall, a decidedly incomplete texture of Yukawa interactions would have been obtained. This is a clear example of why one must be careful to take into account *all* stability walls of the bundle when attempting to determine the Yukawa texture of a heterotic vacuum.

## 6 Textures in a Three Generation Model

### 6.1 One Heavy Family and Other Textures

Previously, we investigated Yukawa textures arising from stability walls of  $SU(3)$  bundles. In this section, we discuss stability walls and their constraint on matter textures in a *more phenomenologically realistic* context. Specifically, we consider the  $SO(10)$  theory associated with an  $SU(4)$  bundle. We further assume that this bundle is destabilized by a single rank two sub-bundle which gives rise to a *single* stability wall in the Kähler cone and a *single* D-term.

In the stable chamber, the structure group of  $V$  is  $SU(4)$  and, hence, the low energy theory has a gauged  $SO(10)$  symmetry. As in previous sections, along the wall of poly-stability the vector bundle splits into a direct sum

$$V \rightarrow \mathcal{F}_1 \oplus \mathcal{F}_2, \quad (6.1)$$

where now both sub-bundles have rank two. The structure group then changes from  $SU(4)$  to  $S[U(2) \times U(2)]$ . Since the commutant of  $S[U(2) \times U(2)]$  in  $E_8$  is  $SO(10) \times U(1)$ , the symmetry

of the four-dimensional theory is enhanced by an anomalous  $U(1)$ . On the stability wall, where  $V$  decomposes as (6.1), the fields carry an extra  $U(1)$  charge in addition to their  $SO(10)$  content. We present the generic zero-mode spectrum in Table 6.

Our goal is to illustrate how the stability wall can constrain Yukawa textures in a phenomenologically realistic context. Therefore, we will only consider bundles leading to *three generations* of chiral matter. To simplify the analysis, these bundles will be further restricted so that the multiplicities of the **16** and **10** fields on the stability wall, and, hence, in any of its branches, are

$$\begin{aligned} n_{16_{+1}} &= h^1(X, \mathcal{F}_1) = 2, & n_{16_{-1}} &= h^1(X, \mathcal{F}_2) = 1 \\ n_{10_{+2}} &= h^1(X, \wedge^2 \mathcal{F}_2) = 1 \end{aligned} \tag{6.2}$$

and no other  $SO(10)$  non-singlets occur. The theory generically contains both  $C_1 \in H^1(X, \mathcal{F}_1 \times \mathcal{F}_2^*)$  with charge  $+2$  as well as  $C_2 \in H^1(X, \mathcal{F}_2 \times \mathcal{F}_1^*)$  with charge  $-2$ . To cubic order in the matter fields, the  $SO(10) \times U(1)$  invariant superpotential is

$$W = h_1 f_2^2 + h_1 f_1 f_2 C_2 + h_1 f_1^2 C_2^2 + (C_1 C_2)^2. \tag{6.3}$$

As always, we ignore irrelevant higher dimension terms. *On the stability wall*, the FI piece of the associated D-term vanishes. To have an  $\mathcal{N} = 1$  supersymmetric Minkowski vacuum, it follows from (2.21) and (6.3) that  $\langle C_1 \rangle = \langle C_2 \rangle = 0$ . Therefore,

$$W_{\text{Yukawa}}^{\text{wall}} = h_1 f_2^2. \tag{6.4}$$

This is a very restrictive Yukawa texture, giving non-vanishing mass to only one matter family. What happens for small deformations away from this wall into a stable chamber?

To do this, one has to specify which of the two rank two sub-bundles in (6.1) destabilizes  $V$ . Let us first choose this to be  $\mathcal{F}_2$ . Then,  $V$  can be constructed from the sequence

$$0 \rightarrow \mathcal{F}_2 \rightarrow V \rightarrow \mathcal{F}_1 \rightarrow 0 \tag{6.5}$$

and it is  $C_2 \in H^1(X, \mathcal{F}_2 \times \mathcal{F}_1^*)$  that controls the movement away from the stability wall into the indecomposable gauge configuration. This can also be seen by inspecting the charges of the various fields in the  $U(1)$  D-term associated with the wall, see (2.21). Since we have chosen  $\mu(\mathcal{F}_2) < 0$  in the stable region, it is the  $C_2$  fields that must acquire a non-zero vev. Furthermore, the  $(C_1 C_2)^2$  term in (6.3) assures that the vevs of  $C_1$  must vanish in the stable region. As a result, the two  $C_2$  field dependent matter couplings in (6.3) “grow back” to contribute to Yukawa couplings near the stability wall. It follows from the holomorphy analysis of subsection 3.2 that one expects

$$W_{\text{Yukawa}} = h_1 f_2^2 + \mathbf{h}_1 \mathbf{f}_1 \mathbf{f}_2 + \mathbf{h}_1 \mathbf{f}_1^2 \tag{6.6}$$

*everywhere* in the interior of the stable chamber for this branch of the vacuum. Note that if the Kähler moduli were stabilized *close to*, but not on, the stability wall, the four-dimensional  $SO(10)$  theory would have one heavy family and a hierarchy for the remaining two generations controlled by powers of  $\langle C_2 \rangle$ .

Let us now consider the second branch where  $\mathcal{F}_1$  is the destabilizing rank two sub-bundle. Then,  $V$  can be constructed from the sequence

$$0 \rightarrow \mathcal{F}_1 \rightarrow V \rightarrow \mathcal{F}_2 \rightarrow 0 \quad (6.7)$$

and it is  $C_1 \in H^1(X, \mathcal{F}_1 \times \mathcal{F}_2^*)$  that controls the movement away from the stability wall into the indecomposable gauge configuration. Since now  $\mu(\mathcal{F}_1) < 0$ , it follows from  $c_1(V) = 0$  that  $\mu(\mathcal{F}_2) = -\mu(\mathcal{F}_1) > 0$ . Hence, the FI term in  $D^{U(1)}$ , which is proportional to  $\mu(\mathcal{F}_2)$ , is positive and it is the  $C_1$  fields that acquire a non-zero vev while the vevs of  $C_2$  vanish. One must then conclude from (6.3) that *no* Yukawa couplings can “grow back” near the stability wall in this branch. It follows from the holomorphy analysis of subsection 3.2 that one expects

$$W_{\text{Yukawa}} = h_1 f_2^2 \quad (6.8)$$

*everywhere* in the interior of the stable chamber for this branch of the vacuum. Therefore, stability wall constraints can provide a natural way of obtaining a single heavy family in heterotic three family vacua.

Representation	Field name	Cohomology
$(\mathbf{1}, \mathbf{2}, \mathbf{2})_2$	$C_1$	$H^1(X, \mathcal{F}_1 \otimes \mathcal{F}_2^*)$
$(\mathbf{1}, \mathbf{2}, \mathbf{2})_{-2}$	$C_2$	$H^1(X, \mathcal{F}_2 \otimes \mathcal{F}_1^*)$
$(\mathbf{1}, \mathbf{3}, \mathbf{1})_0$	$\phi_1$	$H^1(X, \mathcal{F}_1 \otimes \mathcal{F}_1^*)$
$(\mathbf{1}, \mathbf{1}, \mathbf{3})_0$	$\phi_2$	$H^1(X, \mathcal{F}_2 \otimes \mathcal{F}_2^*)$
$(\mathbf{16}, \mathbf{2}, \mathbf{1})_1$	$f_1$	$H^1(X, \mathcal{F}_1)$
$(\mathbf{16}, \mathbf{1}, \mathbf{2})_{-1}$	$f_2$	$H^1(X, \mathcal{F}_2)$
$(\overline{\mathbf{16}}, \mathbf{2}, \mathbf{1})_{-1}$	$\tilde{f}_1$	$H^1(X, \mathcal{F}_1^*)$
$(\overline{\mathbf{16}}, \mathbf{1}, \mathbf{2})_1$	$\tilde{f}_2$	$H^1(X, \mathcal{F}_2^*)$
$(\mathbf{10}, \mathbf{1}, \mathbf{1})_2$	$h_1$	$H^1(X, \wedge^2 \mathcal{F}_1)$
$(\mathbf{10}, \mathbf{1}, \mathbf{1})_{-2}$	$h_2$	$H^1(X, \wedge^2 \mathcal{F}_2)$
$(\mathbf{10}, \mathbf{2}, \mathbf{2})_0$	$h_3$	$H^1(X, \mathcal{F}_1 \otimes \mathcal{F}_2)$

Table 6: The spectrum of a *generic*  $SU(4)$  bundle decomposing into two rank 2 bundles,  $\mathcal{F}_1 \oplus \mathcal{F}_2$ , on the stability wall. The resulting structure group is  $S[U(2) \times U(2)]$ .

## 6.2 An Explicit Three Generation Model

In this subsection, we present an example of a three generation model with the stability wall structure described above and only one heavy family. To begin, consider a vector bundle over a simply connected Calabi-Yau threefold,  $X$ , which admits a fixed-point free, discrete automorphism  $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$ . This discrete symmetry allows one to construct a smooth quotient manifold



$\hat{X} = X/(\mathbb{Z}_3 \times \mathbb{Z}_3)$  that is not simply connected<sup>11</sup>. By choosing a vector bundle  $V$  over the “upstairs” manifold,  $X$ , which admits an equivariant structure under this symmetry, one can create a bundle  $\hat{V}$  on the “downstairs” threefold  $\hat{X}$ . This “quotienting” process is somewhat convoluted mathematically and, since it is not the central focus of this paper, we present here only the spectrum and properties of the final bundle  $\hat{V}$  on  $\hat{X}$ . The derivation of this bundle in terms of its descent from the “upstairs” theory, as well as relevant technical details, are given in Appendix B.

The Calabi-Yau threefold is taken to be a  $\mathbb{Z}_3 \times \mathbb{Z}_3$  quotient of the bi-cubic hypersurface in  $\mathbb{P}^3 \times \mathbb{P}^3$  [18],

$$\hat{X}^{2,11} = X/\mathbb{Z}_3 \times \mathbb{Z}_3, \quad X = \left[ \begin{array}{c|c} \mathbb{P}^2 & 3 \\ \hline \mathbb{P}^2 & 3 \end{array} \right]^{2,83}. \quad (6.9)$$

Before giving a vector bundle on  $\hat{X}$ , we first describe how line bundles on  $\hat{X}$  are related to those on  $X$ . Recall that  $\mathcal{O}(k, m)$  is the line bundle  $\mathcal{O}(kH_1 + mH_2)$ , where  $H_1, H_2$  are the restrictions of the hyperplanes of each  $\mathbb{P}^2$  to  $X$ . The two-dimensional space of divisors of  $X$  is spanned by  $H_i$ ,  $i = 1, 2$ , and we choose this as a convenient basis to describe divisors and the line bundles associated with them. To describe a line bundle on  $\hat{X}$ , note that the basis of ample divisors on  $\hat{X}$  is related to those on  $X$  via the quotient map. Specifically, given the projection map  $q : X \rightarrow \hat{X}$ , a divisor  $\hat{H}$  of  $\hat{X}$  is related to some divisor  $H$  on  $X$  via  $q^*(\hat{H}) = H$ . Using this, we choose a basis of divisors  $\hat{H}_i$ ,  $i = 1, 2$  on  $\hat{X}$  to be related to  $H_i$  via the pull-backs

$$q^*(\hat{H}_1) = 3H_1, \quad q^*(\hat{H}_2) = H_1 + H_2. \quad (6.10)$$

Using the divisor/line bundle correspondence, the basis of Kähler forms of  $\hat{X}$  are then related to those on  $X$  by  $q^*(\hat{J}_1) = 3J_1$  and  $q^*(\hat{J}_2) = J_1 + J_2$ .

We define the rank four  $SU(4)$  vector bundle on  $\hat{X}$  via the exact sequence

$$0 \rightarrow \hat{\mathcal{F}}_1 \rightarrow \hat{V} \rightarrow \hat{\mathcal{F}}_2 \rightarrow 0, \quad (6.11)$$

where  $\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2$  are rank two bundles constructed from

$$\begin{aligned} 0 \rightarrow \hat{\mathcal{F}}_1 \rightarrow \mathcal{Q}_1 \rightarrow \mathcal{O}(2\hat{H}_2) \rightarrow 0, \\ 0 \rightarrow \hat{\mathcal{F}}_2 \rightarrow \mathcal{Q}_2 \rightarrow \mathcal{O}(\hat{H}_2) \rightarrow 0 \end{aligned} \quad (6.12)$$

and  $\mathcal{Q}_1, \mathcal{Q}_2$  are rank three bundles defined via their pull-backs

$$q^*(\mathcal{Q}_1) = \mathcal{O}(0, 2)^{\oplus 3} \quad q^*(\mathcal{Q}_2) = \mathcal{O}(1, -1)^{\oplus 3} \quad (6.13)$$

to sums of line bundles on  $X$ . Since  $c_1(\hat{\mathcal{F}}_1) = (-2, 4)$  and  $c_1(\hat{\mathcal{F}}_2) = (2, -4)$ , then  $c_1(\hat{V}) = 0$  and  $\hat{V}$  in (6.11) defines an  $SU(4)$  bundle over  $\hat{X}$ . The resulting four-dimensional theory has  $SO(10)$  gauge symmetry. The matter spectrum of  $\hat{V}$  is derived in Appendix B and given by

$$\begin{aligned} n_{16} &= h^1(\hat{X}, \hat{V}) = h^1(\hat{X}, \hat{\mathcal{F}}_1) + h^1(\hat{X}, \hat{\mathcal{F}}_2) = 2 + 1 = 3, \\ n_{\bar{16}} &= h^1(\hat{X}, \hat{V}^*) = 0, \\ n_{10} &= h^1(\hat{X}, \wedge^2 \hat{V}) = h^1(\hat{X}, \wedge^2 \hat{\mathcal{F}}_1) = 2, \end{aligned} \quad (6.14)$$

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<sup>11</sup>The first fundamental group of the quotient manifold is  $\pi_1(\hat{X}) = \mathbb{Z}_3 \times \mathbb{Z}_3$ .

which is very similar to (6.2) in the preceding subsection, with the slight exception that there are two **10**'s.

The bundle, (6.11), is not stable everywhere in the Kähler cone. By construction,  $\hat{V}$  is destabilized by the bundle  $\hat{\mathcal{F}}_1$  in some region of Kähler moduli space. The region of stability is shown in Figure 3. This should be compared with the stability wall associated with the “upstairs” bundle on  $X$ , presented in Figure 4 of Appendix B<sup>12</sup>. On the stability wall,  $\hat{V}$  decomposes as  $\hat{V} \rightarrow \hat{\mathcal{F}}_1 \oplus \hat{\mathcal{F}}_2$  and the structure group changes from  $SU(4)$  to  $S[U(2) \times U(2)]$ . The  $SO(10)$  gauge symmetry of the effective theory is then enhanced by an additional  $U(1)$  symmetry to  $SO(10) \times U(1)$ . As a result, the **16** and **10** multiplets of  $SO(10)$ , as well as the bundle moduli, carry an additional charge. The decomposition of the cohomology under the enhanced symmetry group for this *explicit* example is presented in Table 7. This is a subset of the generic spectrum of Table 6. Note that, in addition to the matter multiplicities (6.14), there are 9  $C_1$  type fields. However, no  $C_2$  fields appear. Hence, this example describes the second branch of the generic vacuum discussed above. It follows that

$$W_{\text{Yukawa}} = h_1 f_2^2 \quad (6.15)$$

*everywhere* in the interior of the stable chamber. We conclude that this explicit vacuum naturally

Representation	Field name	Cohomology	Multiplicity
$(\mathbf{1}, \mathbf{2}, \mathbf{2})_2$	$\hat{C}_1$	$H^1(\hat{X}, \hat{\mathcal{F}}_1 \otimes \hat{\mathcal{F}}_2^*)$	9
$(\mathbf{1}, \mathbf{3}, \mathbf{1})_0$	$\hat{\phi}_1$	$H^1(\hat{X}, \hat{\mathcal{F}}_1 \otimes \hat{\mathcal{F}}_1^*)$	1
$(\mathbf{1}, \mathbf{1}, \mathbf{3})_0$	$\hat{\phi}_2$	$H^1(\hat{X}, \hat{\mathcal{F}}_2 \otimes \hat{\mathcal{F}}_2^*)$	1
$(\mathbf{16}, \mathbf{2}, \mathbf{1})_1$	$\hat{f}_1$	$H^1(\hat{X}, \hat{\mathcal{F}}_1)$	2
$(\mathbf{16}, \mathbf{1}, \mathbf{2})_{-1}$	$\hat{f}_2$	$H^1(\hat{X}, \hat{\mathcal{F}}_2)$	1
$(\mathbf{10}, \mathbf{1}, \mathbf{1})_2$	$\hat{h}_1$	$H^1(\hat{X}, \wedge^2 \hat{\mathcal{F}}_1)$	2

Table 7: The “downstairs” field content of the *explicit* bundle decomposition  $\hat{V} \rightarrow \hat{\mathcal{F}}_1 \oplus \hat{\mathcal{F}}_2$  defined by (6.11), (6.12) and (6.13).

has one heavy family within the context of a realistic particle physics model.

## 7 Constraints on Massive Vector-Like Pairs

Extended  $U(1)$  gauge symmetry constrains the superpotential on and near any stability wall and, by holomorphicity, in the interior of each stable chamber in the Kähler cone. So far, we have focused on the implications of this for cubic matter interactions, that is, Yukawa textures. However, the

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<sup>12</sup>The stability wall structure of a bundle  $\hat{V}$  on a quotient manifold is entirely determined by the stability structure of  $V$  on  $X$ . Since only those sub-bundles of  $V$  which are equivariant under the finite group action descend to sub-bundles of  $\hat{V}$  on  $\hat{X}$ , the number of stability walls can at most decrease in going from  $X$  to  $\hat{X}$ .

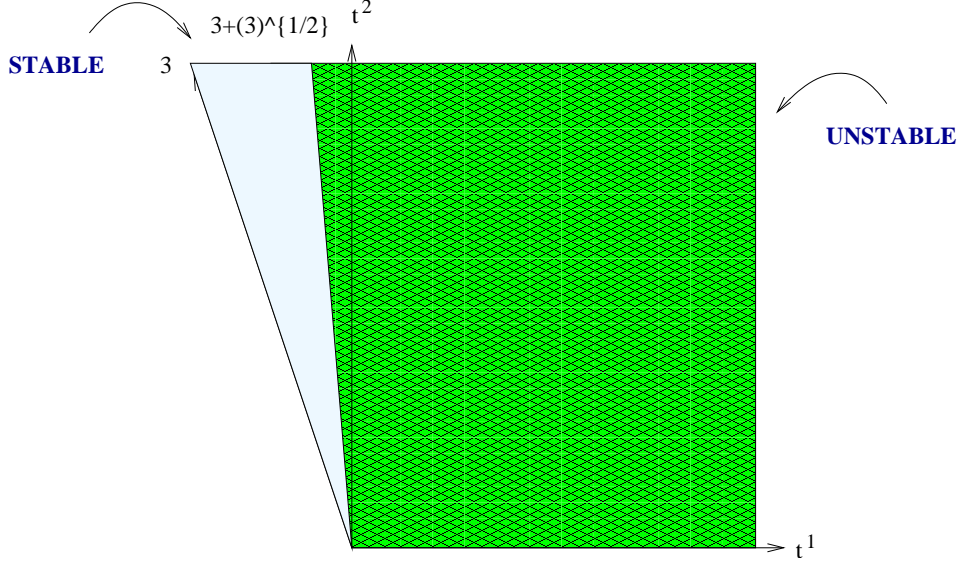


Figure 3: The Kähler cone ( $t^2 > 0$  and  $t^2 + 3t^1 > 0$ ) and the regions of stability/instability for the “downstairs” bundle  $\hat{V} = V/(\mathbb{Z}_3 \times \mathbb{Z}_3)$  on the quotient manifold  $\hat{X} = X/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ , defined respectively by (6.11) and (6.9). At the line with slope  $t^2/t^1 = -3 - \sqrt{3}$ ,  $\hat{V}$  decomposes as  $\hat{V} \rightarrow \hat{\mathcal{F}}_1 \oplus \hat{\mathcal{F}}_2$  given in (6.12) and (6.13).

existence of stability walls constrains all terms in the superpotential, not just Yukawa couplings. In this section, we broaden our analysis to couplings involving vector-like pairs of matter multiplets. We show that extended  $U(1)$  symmetry can forbid many, and sometimes all, such pairs from gaining superpotential mass terms. This can have important implications for heterotic model building.

Generically, the zero-mode spectrum of a bundle on a stability wall arises from the cohomology of the sub-bundles into which it decomposes. In particular, matter can be in both a non-singlet representation and its conjugate representation of the low-energy gauge group. All such matter can occur on the stability wall, their multiplicity depending on the specific vacuum chosen. As one moves away from the wall into a stable chamber, the zero-mode spectrum can change. The Atiyah-Singer index theorem [28] requires that the chiral asymmetry of the matter representations be preserved. For example, for a stable  $SU(3)$  bundle  $V$  which decomposes into  $V = \mathcal{F} \oplus \mathcal{K}$  on the stability wall,

$$h^1(V) - h^1(V^*) = h^1(\mathcal{F}) - h^1(\mathcal{F}^*) + h^1(\mathcal{K}) - h^1(\mathcal{K}^*). \quad (7.1)$$

However, the actual number of matter representations need not stay the same. Specifically, as one moves away from the wall, certain  $U(1)$  charged  $C$  fields get a vev so as to preserve  $\mathcal{N} = 1$  supersymmetry. In principle, these can induce a non-vanishing mass for any vector-like pair of matter representations. As we have already seen, however, the extended  $U(1)$  symmetry imposes serious constraints on cubic, and higher, matter couplings. We expect there to be vector-like “mass texture” as well. As throughout this paper, we find it easiest to analyze vector-like pair masses within the context of explicit examples.

## 7.1 One Wall with One D-Term

Let us first consider the class of vacua discussed in Subsection 2.6 and Section 3. In this case,  $h^{1,1}(X) = 2$  and  $V$  is an  $SU(3)$  bundle which decomposes at a single stability wall into  $V = \mathcal{F} \oplus \mathcal{K}$ , where  $\mathcal{F}$  and  $\mathcal{K}$  have rank one and two respectively. The generic spectrum on the wall arises as the product cohomologies of  $\mathcal{F}$  and  $\mathcal{K}$ , and is labeled by representations of the extended  $E_6 \times U(1)$  four-dimensional gauge group. This is presented in Table 2. The most general gauge invariant superpotential involving terms *cubic* in the  $F$ 's was given in (3.1). We now extend this to include all relevant terms involving  $\mathbf{27} \cdot \overline{\mathbf{27}}$  *vector-like pairs* of matter multiplets. The result is

$$W = \cdots + C_1 F_1 \bar{F}_2 + C_2 F_2 \bar{F}_1 + C_1 C_2 F_1 \bar{F}_1 + C_1 C_2 F_2 \bar{F}_2, \quad (7.2)$$

where terms are shown in order of increasing dimension and we have suppressed all parameters and indices. Note that no quadratic terms appear, since, on the wall, all matter fields are zero-modes. Finally, each term can be multiplied by any positive power of  $C_1 C_2$ . Such terms do not change the subsequent analysis and, in the interest of brevity, we ignore them.

As discussed previously, *on the stability wall* the requirement of  $\mathcal{N} = 1$  supersymmetry and vanishing cosmological constant constrains  $\langle C_1 \rangle = \langle C_2 \rangle = 0$ . It follows from (7.2) that

$$W_{\text{vec-like pairs}}^{\text{wall}} = 0, \quad (7.3)$$

consistent with the fact that  $F_1$ ,  $F_2$  and  $\bar{F}_1$ ,  $\bar{F}_2$  are all zero-modes on the wall. What happens as we move into the interior a stable region? As discussed in Section 2.6, there are two stable branches of moduli space. These are specified by choosing either  $\langle C_1 \rangle = 0, \langle C_2 \rangle \neq 0$ , corresponding to  $\mu(\mathcal{F}) < 0$ , or  $\langle C_1 \rangle \neq 0, \langle C_2 \rangle = 0$ , corresponding to  $\mu(\mathcal{K}) < 0$ . Consider the first branch. In this case, it follows from (7.2) and the holomorphicity of the superpotential that *everywhere* in this chamber of Kähler moduli space

$$W_{\text{vec-like pairs}} = \mathbf{F}_2 \bar{\mathbf{F}}_1. \quad (7.4)$$

Note that the non-zero  $C_2$  vevs have allowed some vector-like mass terms missing in (7.3) to “grow back”. These are expressed in boldface, as were Yukawa couplings that regrew away the wall. We conclude that in the interior of the stable chamber specified by  $\langle C_1 \rangle = 0, \langle C_2 \rangle \neq 0$ , superfields  $F_2$  and  $\bar{F}_1$  appear in non-vanishing mass terms. However, the extended  $U(1)$  gauge symmetry on the stability wall *forbids* vector-like masses for  $F_1$  and  $\bar{F}_2$  from developing. Now consider the second branch. In this case, it follows from (7.2) and holomorphicity that *everywhere* in this stable chamber

$$W_{\text{vec-like pairs}} = \mathbf{F}_1 \bar{\mathbf{F}}_2. \quad (7.5)$$

Hence, in the interior of the stable chamber specified by  $\langle C_1 \rangle \neq 0, \langle C_2 \rangle = 0$ , the extended  $U(1)$  gauge symmetry on the stability wall, while allowing superfields  $F_1$  and  $\bar{F}_2$  to appear in mass terms, *forbids* vector-like masses for  $F_2$  and  $\bar{F}_1$ .

This is a clear example where the stable chambers next to a stability wall exhibit non-trivial *vector-like mass textures*; allowing some mass terms while forbidding others.

## 7.2 One Wall with Two D-Terms

We now move on to consider the class of vacua discussed in Section 4. In this case,  $h^{1,1}(X) = 2$  and  $V$  is an  $SU(3)$  bundle which decomposes at a single stability wall into  $V = l_1 \oplus l_2 \oplus l_3$ , where  $l_i$ ,  $i = 1, 2, 3$  are line bundles. The generic spectrum on the wall arises as the product cohomologies of  $l_1, l_2, l_3$  and is labeled by representations of the extended  $E_6 \times U(1) \times U(1)$  four-dimensional gauge group. This is presented in Table 3. The most general gauge invariant superpotential involving *cubic* couplings in the  $F$ 's was given in (4.10). We now extend this result to include all relevant terms involving  $\mathbf{27} \cdot \overline{\mathbf{27}}$  *vector-like pairs* of matter multiplets. The result is

$$\begin{aligned} W = \dots &+ C_1 f_1 \tilde{f}_2 + C_2 f_1 \tilde{f}_3 + \tilde{C}_1 f_2 \tilde{f}_1 + \tilde{C}_3 f_2 \tilde{f}_3 + \tilde{C}_2 f_3 \tilde{f}_1 + C_3 f_3 \tilde{f}_2 \\ &+ \left( C_1 \tilde{C}_1 + C_2 \tilde{C}_2 + C_3 \tilde{C}_3 \right) \left( f_1 \tilde{f}_1 + f_2 \tilde{f}_2 + f_3 \tilde{f}_3 \right) \\ &+ C_1 \tilde{C}_2 f_3 \tilde{f}_2 + C_1 \tilde{C}_3 f_1 \tilde{f}_3 + \tilde{C}_1 C_2 f_2 \tilde{f}_3 + \tilde{C}_1 C_3 f_3 \tilde{f}_1 + C_2 C_3 f_1 \tilde{f}_2 + \tilde{C}_2 \tilde{C}_3 f_2 \tilde{f}_1 \end{aligned} \quad (7.6)$$

No quadratic terms appear since all superfields are zero-modes on the wall. We have only indicated terms involving at most two different  $C$  fields. Vevs of the product of three or more different  $C$  fields must necessarily vanish in any branch. Finally, each term can be multiplied by any positive integer power of neutral combinations of  $C$  fields. Such terms do not change the subsequent analysis.

*On the stability wall*, the requirement of supersymmetry and vanishing cosmological constant constrains the vevs of each  $C$  field to vanish. Hence,

$$W_{\text{vec-like pairs}}^{\text{wall}} = 0, \quad (7.7)$$

consistent with the fact that all  $f$  and  $\tilde{f}$  matter fields are zero-modes on the wall. What happens as we move into a stable region? As discussed in Section 4, there are *six* stable branches of the moduli space. Each branch is specified by a different pair  $(\langle C_i \rangle, \langle C_j \rangle)$ ,  $(\langle C_i \rangle, \langle \tilde{C}_j \rangle)$  or  $(\langle \tilde{C}_i \rangle, \langle \tilde{C}_j \rangle)$  being non-vanishing, with all remaining vevs zero. To be specific, let us choose the branch defined by  $\langle \tilde{C}_2 \rangle \neq 0$ ,  $\langle C_3 \rangle \neq 0$ . It then follows from (7.3) and holomorphicity that in the interior of this branch of Kähler moduli space

$$W_{\text{vec-like pairs}} = \mathbf{f}_3 \tilde{\mathbf{f}}_1 + \mathbf{f}_3 \tilde{\mathbf{f}}_2. \quad (7.8)$$

Note that the non-zero  $\tilde{C}_2, C_3$  vevs have allowed some vector-like mass terms missing in (7.7) to “grow back”. Therefore, in the interior of the stable chamber specified by  $\langle \tilde{C}_2 \rangle \neq 0$ ,  $\langle C_3 \rangle \neq 0$ , matter multiplets  $f_3, \tilde{f}_1$  and  $\tilde{f}_2$  appear in non-vanishing mass terms. However, the two extended  $U(1)$  gauge symmetries on the stability wall *forbid* vector-like masses for  $\tilde{f}_3, f_1$  and  $f_2$  from developing.

We conclude that the extended  $U(1)$  gauge symmetries on stability walls in the Kähler cone can lead to restrictive *vector-like mass textures*. Generically, these textures can disallow some vector-like pairs from having a superpotential mass term, a restriction of consequence for phenomenology. Hence, when building realistic smooth heterotic models, it is essential to include all stability walls and their associated constraints in the analysis. This makes theories with only chiral matter appear much more attractive from this perspective.

## 8 Conclusions

In previous work [53, 15, 16], “stability walls”, that is, boundaries separating regions in Kähler moduli space where a non-Abelian internal gauge bundle either preserves or breaks supersymmetry, were explored. The four-dimensional effective theories valid near such boundaries provide us with an explicit low-energy description of the supersymmetry breaking associated with vector bundle slope stability. The central feature of a stability wall is that, near such a locus in moduli space, the internal gauge bundle decomposes into a direct sum and, as a result, the four-dimensional effective theory is enhanced by at least one Green-Schwarz anomalous  $U(1)$  symmetry.

In this paper, we have used this effective theory to investigate the structure and properties of heterotic theories with stability induced sub-structure in their Kähler cones. Specifically, we have used the theory near the stability wall, with its enhanced  $U(1)$  symmetries, to constrain the form of the  $\mathcal{N} = 1$  superpotential  $W$ . Using the fact that the superpotential is a holomorphic function, it is possible to extend these constraints throughout the entire moduli space. As a result, deep into the stable regions of the Kähler cone, where supersymmetric heterotic compactifications are normally considered, strong constraints on the superpotential still persist. Without knowledge of the global supersymmetric properties of the vector bundle (that is, a full understanding of its slope stability), these important textures would be inexplicable or, more seriously, go unnoticed if the Yukawa couplings were not explicitly computed. We would like to point out that some of the couplings that are disallowed in the perturbative textures discussed in this paper may well be reintroduced by non-perturbative effects, such as membrane instantons [61, 62]. Such couplings would be hierarchically smaller than those present perturbatively. This interesting possibility, which also is strongly constrained by the additional Abelian symmetries on the stability walls, will be addressed in a future publication.

We stress again that the existence of stability walls, and their consequences, are the generic situation for a heterotic compactification. In most cases, vector bundles that are slope stable somewhere in moduli space are not slope stable for all polarizations. Hence, the constraints described in this paper must be considered to have a full understanding of the effective theory. Indeed, all three of the main methods of bundle construction in the heterotic literature— monad bundles [20, 21, 19, 26, 18], bundles defined by extension [28, 14] and the spectral cover construction [31, 32, 33] - typically exhibit stability walls. Thus, from the point of view of model building and string phenomenology, the textures and constraints on Yukawa couplings and vector - like masses discussed in this paper are generically present. They must be taken into account in any attempt to build physically realistic models.

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## A All Textures from a Single D-term Stability Wall

In this Appendix, we present all Yukawa textures that can result from holomorphic vector bundles with a *single* stability wall where the bundle splits into a direct sum of *two factors*. These are an important sub-class of Yukawa textures that can appear naturally within the context of heterotic string and M-theory.

### A.1 An $SU(3)$ Bundle with a Stability Wall and One D-Term

We begin with a compactification of heterotic theory on a Calabi-Yau threefold with a rank *three* holomorphic vector bundle  $V$ . For any Kähler form in the stable chamber, the structure group is an indecomposable  $SU(3)$  leading to an  $E_6$  gauge group in the low-energy theory. At the stability wall, where the bundle splits into two parts, an  $SU(3)$  bundle necessarily breaks into a rank 2 and a rank 1 piece, which we will denote by  $\mathcal{F}$  and  $\mathcal{K}$  respectively. That is,

$$V = \mathcal{F} \oplus \mathcal{K} . \quad (\text{A.1})$$

The structure group of this bundle is  $S[U(2) \times U(1)] \cong SU(2) \times U(1)$ , leading to an enhanced  $E_6 \times U(1)$  gauge group in the effective theory. It follows that the relevant decomposition of the **248** of  $E_8$  is

$$E_8 \supset E_6 \times SU(2) \times U(1) \quad (\text{A.2})$$

$$\begin{aligned} \mathbf{248} = & (\mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{2})_3 + (\mathbf{1}, \mathbf{2})_{-3} + (\mathbf{1}, \mathbf{3})_0 + (\mathbf{78}, \mathbf{1})_0 \\ & + (\mathbf{27}, \mathbf{1})_{-2} + (\mathbf{27}, \mathbf{2})_1 + (\overline{\mathbf{27}}, \mathbf{1})_2 + (\overline{\mathbf{27}}, \mathbf{2})_{-1} . \end{aligned} \quad (\text{A.3})$$

This decomposition indicates which representations of  $E_6 \times U(1)$  can possibly appear as fields in the four-dimensional effective theory. To find out how many of each multiplet is actually present, one must calculate the dimension of the cohomology groups indicated in Table 8. As discussed in Section 3, only one of the two fields  $C_1, C_2$  can get a vev. If  $\langle C_1 \rangle \neq 0$ , then the allowed Yukawa couplings are

$$W_{\text{Yukawa}} = f_1 f_2^2 + \tilde{f}_1 \tilde{f}_2^2 + \mathbf{f}_1^3 + \mathbf{f}_1^2 \mathbf{f}_2 + \tilde{\mathbf{f}}_2^3 . \quad (\text{A.4})$$

As discussed in the text, we suppress the arbitrary coefficients in front of each term for simplicity. Terms allowed in the dimension three (in superfields) superpotential on the stability wall are shown in standard type. Yukawa terms that originate as higher dimensional operators involving powers of  $C_1$ , which are “grown back” upon re-entering the interior supersymmetric region where  $\langle C_1 \rangle \neq 0$ , are indicated in boldface. On the other hand, if  $\langle C_2 \rangle \neq 0$ , then we find

$$W_{\text{Yukawa}} = f_1 f_2^2 + \tilde{f}_1 \tilde{f}_2^2 + \tilde{\mathbf{f}}_1^3 + \tilde{\mathbf{f}}_1^2 \tilde{\mathbf{f}}_2 + \mathbf{f}_2^3 . \quad (\text{A.5})$$

Representation	Field name	Cohomology
$(\mathbf{1}, \mathbf{2})_3$	$C_1$	$h^1(X, \mathcal{F}^* \otimes \mathcal{K})$
$(\mathbf{1}, \mathbf{2})_{-3}$	$C_2$	$h^1(X, \mathcal{F} \otimes \mathcal{K}^*)$
$(\mathbf{1}, \mathbf{3})_0$	$\phi$	$h^1(X, \mathcal{F}^* \otimes \mathcal{F})$
$(\mathbf{27}, \mathbf{1})_{-2}$	$f_1$	$h^1(X, \mathcal{K})$
$(\mathbf{27}, \mathbf{2})_1$	$f_2$	$h^1(X, \mathcal{F})$
$(\overline{\mathbf{27}}, \mathbf{1})_2$	$\tilde{f}_1$	$h^1(X, \mathcal{K}^*)$
$(\overline{\mathbf{27}}, \mathbf{2})_{-1}$	$\tilde{f}_2$	$h^1(X, \mathcal{F}^*)$

Table 8:  $SU(3)$  one D-term.

Note that on the stability wall the  $U(1)$  charges strongly restrict the allowed Yukawa couplings. As one moves away from the stability wall into the stable chamber, a number of previously disallowed couplings “grow back”. However, not all terms allowed by the  $E_6$  symmetry can reappear. For example, an  $\tilde{\mathbf{f}}_1^3$  term can never be generated in superpotential (A.4).

## A.2 An $SU(4)$ Bundle with a Stability Wall and One D-Term

Now consider a compactification of heterotic theory on a Calabi-Yau threefold with a rank *four* holomorphic vector bundle  $V$ . For any Kähler form in the stable chamber, the structure group is an indecomposable  $SU(4)$  leading to an  $SO(10)$  gauge group in the low-energy theory. There are now two ways in which the bundle can split at a stability wall. We treat each case in turn. We emphasize that both cases can be realized by rank four bundles with a single stability wall.

### A.2.1 Case 1

The first of the two cases corresponds to the bundle splitting into a rank 3 and a rank 1 piece, which we shall denote by  $\mathcal{F}$  and  $\mathcal{K}$  respectively. That is,

$$V = \mathcal{F} \oplus \mathcal{K} . \quad (\text{A.6})$$

The structure group of this bundle is  $SU(3) \times U(1)$ , leading to an enhanced  $SO(10) \times U(1)$  gauge group in the effective theory. The relevant group theory here is

$$E_8 \supset SO(10) \times SU(3) \times U(1) \quad (\text{A.7})$$

$$\begin{aligned} \mathbf{248} = & (\mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3})_{-4} + (\mathbf{1}, \overline{\mathbf{3}})_4 + (\mathbf{1}, \mathbf{8})_0 + (\mathbf{45}, \mathbf{1})_0 + (\mathbf{16}, \mathbf{1})_3 + (\mathbf{16}, \mathbf{3})_{-1} \\ & + (\overline{\mathbf{16}}, \mathbf{1})_{-3} + (\overline{\mathbf{16}}, \overline{\mathbf{3}})_1 + (\mathbf{10}, \mathbf{3})_2 + (\mathbf{10}, \overline{\mathbf{3}})_{-2} . \end{aligned} \quad (\text{A.8})$$

This decomposition indicates which representations of  $SO(10) \times U(1)$  can possibly appear as fields in the four-dimensional effective theory. To find out how many of each multiplet is actually present,



Representation	Field name	Cohomology
$(\mathbf{1}, \mathbf{3})_{-4}$	$C_1$	$h^1(X, \mathcal{F} \otimes \mathcal{K}^*)$
$(\mathbf{1}, \overline{\mathbf{3}})_4$	$C_2$	$h^1(X, \mathcal{F}^* \otimes \mathcal{K})$
$(\mathbf{1}, \mathbf{8})_0$	$\phi$	$h^1(X, \mathcal{F}^* \otimes \mathcal{F})$
$(\mathbf{16}, \mathbf{1})_3$	$f_1$	$h^1(X, \mathcal{K})$
$(\mathbf{16}, \mathbf{3})_{-1}$	$f_2$	$h^1(X, \mathcal{F})$
$(\overline{\mathbf{16}}, \mathbf{1})_{-3}$	$\tilde{f}_1$	$h^1(X, \mathcal{K}^*)$
$(\overline{\mathbf{16}}, \overline{\mathbf{3}})_1$	$\tilde{f}_2$	$h^1(X, \mathcal{F}^*)$
$(\mathbf{10}, \mathbf{3})_2$	$h_1$	$h^1(X, \mathcal{F} \otimes \mathcal{K})$
$(\mathbf{10}, \overline{\mathbf{3}})_{-2}$	$h_2$	$h^1(X, \wedge^2 \mathcal{F})$

Table 9:  $SU(4)$  one D-term case 1.

one must calculate the dimension of the cohomology groups indicated in Table 9. As discussed previously, only one of the two  $C_1, C_2$  fields can have a non-zero vev. If  $\langle C_1 \rangle \neq 0$ , then the allowed Yukawa couplings are

$$W_{\text{Yukawa}} = h_1 f_2^2 + h_1 \tilde{f}_1 \tilde{f}_2 + h_2 \tilde{f}_2^2 + \mathbf{h}_1 \mathbf{f}_1^2 + \mathbf{h}_1 \mathbf{f}_1 \mathbf{f}_2 + \mathbf{h}_2 \mathbf{f}_1^2 + \mathbf{h}_2 \mathbf{f}_2 \mathbf{f}_2 + \mathbf{h}_1 \tilde{\mathbf{f}}_2^2 . \quad (\text{A.9})$$

On the other hand, if  $\langle C_2 \rangle \neq 0$ , then we find

$$W_{\text{Yukawa}} = h_1 f_2^2 + h_1 \tilde{f}_1 \tilde{f}_2 + h_2 \tilde{f}_2^2 + \mathbf{h}_2 \mathbf{f}_1 \mathbf{f}_2 + \mathbf{h}_2 \mathbf{f}_2^2 + \mathbf{h}_2 \tilde{\mathbf{f}}_1 \tilde{\mathbf{f}}_2 + \mathbf{h}_1 \tilde{\mathbf{f}}_1^2 + \mathbf{h}_2 \tilde{\mathbf{f}}_1^2 . \quad (\text{A.10})$$

### A.2.2 Case 2

The second  $SU(4)$  case corresponds to the rank 4 bundle splitting into two rank 2 pieces denoted by  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . That is,

$$V = \mathcal{F}_1 \oplus \mathcal{F}_2 . \quad (\text{A.11})$$

Under this splitting the structure group breaks into  $SU(2) \times SU(2) \times U(1)$ , leading to an enhanced  $SO(10) \times U(1)$  gauge group in the effective theory. The relevant branchings are

$$E_8 \supset SO(10) \times SU(2) \times SU(2) \times U(1) \quad (\text{A.12})$$

$$\begin{aligned}
\mathbf{248} &= (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1}, \mathbf{3})_0 + (\mathbf{1}, \mathbf{2}, \mathbf{2})_2 + (\mathbf{1}, \mathbf{2}, \mathbf{2})_{-2} + (\mathbf{45}, \mathbf{1}, \mathbf{1})_0 \\
&+ (\mathbf{16}, \mathbf{2}, \mathbf{1})_1 + (\mathbf{16}, \mathbf{1}, \mathbf{2})_{-1} + (\overline{\mathbf{16}}, \mathbf{2}, \mathbf{1})_{-1} + (\overline{\mathbf{16}}, \mathbf{1}, \mathbf{2})_1 + (\mathbf{10}, \mathbf{1}, \mathbf{1})_2 + (\mathbf{10}, \mathbf{1}, \mathbf{1})_{-2} \\
&+ (\mathbf{10}, \mathbf{2}, \mathbf{2})_0 .
\end{aligned} \quad (\text{A.13})$$

The cohomology associated with each field is given in Table 10. As before, only one of the  $C_1, C_2$  fields can get a vev. Which vev is non-zero determines the structure of the Yukawa couplings. For

Representation	Field name	Cohomology
$(\mathbf{1}, \mathbf{2}, \mathbf{2})_2$	$C_1$	$h^1(X, \mathcal{F}_1 \otimes \mathcal{F}_2^*)$
$(\mathbf{1}, \mathbf{2}, \mathbf{2})_{-2}$	$C_2$	$h^1(X, \mathcal{F}_2 \otimes \mathcal{F}_1^*)$
$(\mathbf{1}, \mathbf{3}, \mathbf{1})_0$	$\phi_1$	$h^1(X, \mathcal{F}_1 \otimes \mathcal{F}_1^*)$
$(\mathbf{1}, \mathbf{1}, \mathbf{3})_0$	$\phi_2$	$h^1(X, \mathcal{F}_2 \otimes \mathcal{F}_2^*)$
$(\mathbf{16}, \mathbf{2}, \mathbf{1})_1$	$f_1$	$h^1(X, \mathcal{F}_1)$
$(\mathbf{16}, \mathbf{1}, \mathbf{2})_{-1}$	$f_2$	$h^1(X, \mathcal{F}_2)$
$(\overline{\mathbf{16}}, \mathbf{2}, \mathbf{1})_{-1}$	$\tilde{f}_1$	$h^1(X, \mathcal{F}_1^*)$
$(\overline{\mathbf{16}}, \mathbf{1}, \mathbf{2})_1$	$\tilde{f}_2$	$h^1(X, \mathcal{F}_2^*)$
$(\mathbf{10}, \mathbf{1}, \mathbf{1})_2$	$h_1$	$h^1(X, \wedge^2 \mathcal{F}_1)$
$(\mathbf{10}, \mathbf{1}, \mathbf{1})_{-2}$	$h_2$	$h^1(X, \wedge^2 \mathcal{F}_2)$
$(\mathbf{10}, \mathbf{2}, \mathbf{2})_0$	$h_3$	$h^1(X, \mathcal{F}_1 \otimes \mathcal{F}_2)$

Table 10:  $SU(4)$  one D-term case 2.

$\langle C_1 \rangle \neq 0$ , we find that

$$\begin{aligned}
W_{\text{Yukawa}} = & h_1 f_2^2 + h_2 f_1^2 + h_3 f_1 f_2 + h_1 \tilde{f}_1^2 + h_2 \tilde{f}_2^2 + h_3 \tilde{f}_1 \tilde{f}_2 \\
& + \mathbf{h}_2 \mathbf{f}_1 \mathbf{f}_2 + \mathbf{h}_2 \mathbf{f}_2^2 + \mathbf{h}_3 \mathbf{f}_2^2 + \mathbf{h}_2 \tilde{\mathbf{f}}_1^2 + \mathbf{h}_2 \tilde{\mathbf{f}}_1 \tilde{\mathbf{f}}_2 + \mathbf{h}_3 \tilde{\mathbf{f}}_1^2,
\end{aligned} \tag{A.14}$$

whereas for  $\langle C_2 \rangle \neq 0$  the following texture appears

$$\begin{aligned}
W_{\text{Yukawa}} = & h_1 f_2^2 + h_2 f_1^2 + h_3 f_1 f_2 + h_1 \tilde{f}_1^2 + h_2 \tilde{f}_2^2 + h_3 \tilde{f}_1 \tilde{f}_2 \\
& + \mathbf{h}_1 \mathbf{f}_1 \mathbf{f}_2 + \mathbf{h}_1 \mathbf{f}_1^2 + \mathbf{h}_3 \mathbf{f}_1^2 + \mathbf{h}_1 \tilde{\mathbf{f}}_1 \tilde{\mathbf{f}}_2 + \mathbf{h}_1 \tilde{\mathbf{f}}_2^2 + \mathbf{h}_3 \tilde{\mathbf{f}}_2^2.
\end{aligned} \tag{A.15}$$

### A.3 An $SU(5)$ Bundle with a Stability Wall and One D-Term

Consider compactification of heterotic theory on a Calabi-Yau threefold with a rank *five* holomorphic vector bundle  $V$ . For any Kähler form in the stable chamber, the structure group is an indecomposable  $SU(5)$  leading to an  $SU(5)$  gauge group in the low-energy theory. As in the  $SU(4)$  case, there are two ways in which such bundles can split at a stability wall. We treat these sequentially.

#### A.3.1 Case 1

First consider the case where the bundle splits into a rank 4 and a rank 1 piece, denote by  $\mathcal{F}$  and  $\mathcal{K}$  respectively. That is,

$$V = \mathcal{F} \oplus \mathcal{K}. \tag{A.16}$$

The structure group of this bundle is  $SU(4) \times U(1)$ , leading to an enhanced  $SU(5) \times U(1)$  gauge group in the effective theory. The relevant branching of the **248** representation here is

$$E_8 \supset SU(5) \times SU(4) \times U(1) \quad (\text{A.17})$$

$$\begin{aligned} \mathbf{248} = & (\mathbf{24}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{4})_{-5} + (\mathbf{1}, \bar{\mathbf{4}})_5 + (\mathbf{1}, \mathbf{15})_0 + (\mathbf{10}, \mathbf{1})_4 + (\mathbf{10}, \mathbf{4})_{-1} \\ & + (\bar{\mathbf{10}}, \mathbf{1})_{-4} + (\bar{\mathbf{10}}, \bar{\mathbf{4}})_1 + (\mathbf{5}, \bar{\mathbf{4}})_{-3} + (\mathbf{5}, \bar{\mathbf{6}})_2 + (\bar{\mathbf{5}}, \mathbf{4})_3 + (\bar{\mathbf{5}}, \mathbf{6})_{-2} . \end{aligned} \quad (\text{A.18})$$

The multiplicities of the matter multiplets, in terms of cohomologies of  $\mathcal{F}$  and  $\mathcal{K}$ , may be found in Table 11. As in the previous cases, there are two possible textures. For  $\langle C_1 \rangle \neq 0$ , we find that

Representation	Field name	Cohomology
$(\mathbf{1}, \mathbf{4})_{-5}$	$C_1$	$h^1(X, \mathcal{F} \otimes \mathcal{K}^*)$
$(\mathbf{1}, \bar{\mathbf{4}})_5$	$C_2$	$h^1(X, \mathcal{F}^* \otimes \mathcal{K})$
$(\mathbf{1}, \mathbf{15})_0$	$\phi$	$h^1(X, \mathcal{F}^* \otimes \mathcal{F})$
$(\mathbf{10}, \mathbf{1})_4$	$f_1$	$h^1(X, \mathcal{K})$
$(\mathbf{10}, \mathbf{4})_{-1}$	$f_2$	$h^1(X, \mathcal{F})$
$(\bar{\mathbf{10}}, \mathbf{1})_{-4}$	$\tilde{f}_1$	$h^1(X, \mathcal{K}^*)$
$(\bar{\mathbf{10}}, \bar{\mathbf{4}})_1$	$\tilde{f}_2$	$h^1(X, \mathcal{F}^*)$
$(\mathbf{5}, \bar{\mathbf{4}})_{-3}$	$h_1$	$h^1(X, \mathcal{F}^* \otimes \mathcal{K}^*)$
$(\mathbf{5}, \bar{\mathbf{6}})_2$	$h_2$	$h^1(X, \wedge^2 \mathcal{F}^*)$
$(\bar{\mathbf{5}}, \mathbf{4})_3$	$\tilde{h}_1$	$h^1(X, \mathcal{F} \otimes \mathcal{K})$
$(\bar{\mathbf{5}}, \mathbf{6})_{-2}$	$\tilde{h}_2$	$h^1(X, \wedge^2 \mathcal{F})$

Table 11:  $SU(5)$  one D-term case 1.

$$\begin{aligned} W_{\text{Yukawa}} = & h_1 f_1 f_2 + h_2 f_2^2 + \mathbf{h}_1 \mathbf{f}_1^2 + \mathbf{h}_2 \mathbf{f}_1^2 + \mathbf{h}_2 \mathbf{f}_1 \mathbf{f}_2 \\ & + \tilde{h}_1 \tilde{f}_1 \tilde{f}_2 + \tilde{h}_2 \tilde{f}_2^2 + \tilde{\mathbf{h}}_1 \tilde{\mathbf{f}}_2^2 \\ & + f_1 \tilde{h}_2^2 + f_2 \tilde{h}_1 \tilde{h}_2 + \mathbf{f}_1 \tilde{\mathbf{h}}_1^2 + \mathbf{f}_1 \tilde{\mathbf{h}}_1 \tilde{\mathbf{h}}_2 + \mathbf{f}_2 \tilde{\mathbf{h}}_1^2 \\ & + \tilde{f}_1 h_2^2 + \tilde{f}_2 h_1 h_2 + \tilde{\mathbf{f}}_2 \mathbf{h}_2^2 , \end{aligned} \quad (\text{A.19})$$

where we have grouped all of the  $\mathbf{5} \cdot \mathbf{10} \cdot \mathbf{10}$ ,  $\bar{\mathbf{5}} \cdot \bar{\mathbf{10}} \cdot \bar{\mathbf{10}}$ ,  $\mathbf{10} \cdot \bar{\mathbf{5}} \cdot \bar{\mathbf{5}}$  and  $\bar{\mathbf{10}} \cdot \mathbf{5} \cdot \mathbf{5}$  couplings together on different lines. For  $\langle C_2 \rangle \neq 0$ , the following texture appears

$$\begin{aligned} W_{\text{Yukawa}} = & h_1 f_1 f_2 + h_2 f_2^2 + \mathbf{h}_1 \mathbf{f}_2^2 \\ & + \tilde{h}_1 \tilde{f}_1 \tilde{f}_2 + \tilde{h}_2 \tilde{f}_2^2 + \tilde{\mathbf{h}}_1 \tilde{\mathbf{f}}_1^2 + \tilde{\mathbf{h}}_2 \tilde{\mathbf{f}}_1^2 + \tilde{\mathbf{h}}_2 \tilde{\mathbf{f}}_1 \tilde{\mathbf{f}}_2 \\ & + f_1 \tilde{h}_2^2 + f_2 \tilde{h}_1 \tilde{h}_2 + \mathbf{f}_2 \tilde{\mathbf{h}}_2^2 \\ & + \tilde{f}_1 h_2^2 + \tilde{f}_2 h_1 h_2 + \tilde{\mathbf{f}}_1 \mathbf{h}_1^2 + \tilde{\mathbf{f}}_1 \mathbf{h}_1 \mathbf{h}_2 + \tilde{\mathbf{f}}_2 \mathbf{h}_1^2 . \end{aligned} \quad (\text{A.20})$$

### A.3.2 Case 2

Second, a rank 5 bundle can split into a rank 3 and a rank 2 piece,  $\mathcal{G}$  and  $\mathcal{F}$  respectively, at the stability wall. That is,

$$V = \mathcal{G} \oplus \mathcal{F} . \quad (\text{A.21})$$

The structure group of this bundle is  $SU(2) \times SU(3) \times U(1)$ , leading to an enhanced  $SU(5) \times U(1)$  gauge group in the effective theory. The relevant branching of the **248** representation is given by

$$E_8 \supset SU(5) \times SU(2) \times SU(3) \times U(1) \quad (\text{A.22})$$

$$\begin{aligned} \mathbf{248} = & (\mathbf{24}, \mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{2}, \mathbf{3})_{-5} + (\mathbf{1}, \mathbf{2}, \bar{\mathbf{3}})_5 + (\mathbf{1}, \mathbf{1}, \mathbf{8})_0 \\ & + (\mathbf{10}, \mathbf{2}, \mathbf{1})_3 + (\mathbf{10}, \mathbf{1}, \mathbf{3})_{-2} + (\bar{\mathbf{10}}, \mathbf{2}, \mathbf{1})_{-3} + (\bar{\mathbf{10}}, \mathbf{1}, \bar{\mathbf{3}})_2 + (\mathbf{5}, \mathbf{1}, \mathbf{1})_{-6} + (\mathbf{5}, \mathbf{1}, \mathbf{3})_4 \\ & + (\mathbf{5}, \mathbf{2}, \bar{\mathbf{3}})_{-1} + (\bar{\mathbf{5}}, \mathbf{1}, \mathbf{1})_6 + (\bar{\mathbf{5}}, \mathbf{1}, \bar{\mathbf{3}})_{-4} + (\bar{\mathbf{5}}, \mathbf{2}, \mathbf{3})_1 \end{aligned} \quad (\text{A.23})$$

The multiplicities of each representation, as seen in four dimensions, are given in Table 12. As

Representation	Field name	Cohomology
$(\mathbf{1}, \mathbf{2}, \mathbf{3})_{-5}$	$C_1$	$h^1(X, \mathcal{F}^* \otimes \mathcal{G})$
$(\mathbf{1}, \mathbf{2}, \bar{\mathbf{3}})_5$	$C_2$	$h^1(X, \mathcal{F} \otimes \mathcal{G}^*)$
$(\mathbf{1}, \mathbf{3}, \mathbf{1})_0$	$\phi_1$	$h^1(X, \mathcal{F} \otimes \mathcal{F}^*)$
$(\mathbf{1}, \mathbf{1}, \mathbf{8})_0$	$\phi_2$	$h^1(X, \mathcal{G} \otimes \mathcal{G}^*)$
$(\mathbf{10}, \mathbf{2}, \mathbf{1})_3$	$f_1$	$h^1(X, \mathcal{F})$
$(\mathbf{10}, \mathbf{1}, \mathbf{3})_{-2}$	$f_2$	$h^1(X, \mathcal{G})$
$(\bar{\mathbf{10}}, \mathbf{2}, \mathbf{1})_{-3}$	$\tilde{f}_1$	$h^1(X, \mathcal{F}^*)$
$(\bar{\mathbf{10}}, \mathbf{1}, \bar{\mathbf{3}})_2$	$\tilde{f}_2$	$h^1(X, \mathcal{G}^*)$
$(\mathbf{5}, \mathbf{1}, \mathbf{1})_{-6}$	$h_1$	$h^1(X, \wedge^2 \mathcal{F}^*)$
$(\mathbf{5}, \mathbf{1}, \mathbf{3})_4$	$h_2$	$h^1(X, \wedge^2 \mathcal{G}^*)$
$(\mathbf{5}, \mathbf{2}, \bar{\mathbf{3}})_{-1}$	$h_3$	$h^1(X, \mathcal{G}^* \otimes \mathcal{F}^*)$
$(\bar{\mathbf{5}}, \mathbf{1}, \mathbf{1})_6$	$\tilde{h}_1$	$h^1(X, \wedge^2 \mathcal{F})$
$(\bar{\mathbf{5}}, \mathbf{1}, \bar{\mathbf{3}})_{-4}$	$\tilde{h}_2$	$h^1(X, \wedge^2 \mathcal{G})$
$(\bar{\mathbf{5}}, \mathbf{2}, \mathbf{3})_1$	$\tilde{h}_3$	$h^1(X, \mathcal{F} \otimes \mathcal{G})$

Table 12:  $SU(5)$  one D-term case 2.

always, only one of  $C_1$ ,  $C_2$  can have a non-zero vev. This leads us to two different textures. For  $\langle C_1 \rangle \neq 0$ , we find that

$$\begin{aligned}
W_{\text{Yukawa}} = & h_1 f_1^2 + h_2 f_2^2 + h_3 f_1 f_2 + \mathbf{h}_2 \mathbf{f}_1^2 + \mathbf{h}_2 \mathbf{f}_1 \mathbf{f}_2 + \mathbf{h}_3 \mathbf{f}_1^2 \\
& + \tilde{h}_1 \tilde{f}_1^2 + \tilde{h}_2 \tilde{f}_2^2 + \tilde{h}_3 \tilde{f}_1 \tilde{f}_2 + \tilde{\mathbf{h}}_1 \tilde{\mathbf{f}}_1 \tilde{\mathbf{f}}_2 + \tilde{\mathbf{h}}_1 \tilde{\mathbf{f}}_2^2 + \tilde{\mathbf{h}}_3 \tilde{\mathbf{f}}_2^2 \\
& + f_1 \tilde{h}_2 \tilde{h}_3 + f_2 \tilde{h}_1 \tilde{h}_2 + f_2 \tilde{h}_3^2 + \mathbf{f}_1 \tilde{\mathbf{h}}_1^2 + \mathbf{f}_1 \tilde{\mathbf{h}}_1 \tilde{\mathbf{h}}_2 + \mathbf{f}_1 \tilde{\mathbf{h}}_1 \tilde{\mathbf{h}}_3 + \mathbf{f}_1 \tilde{\mathbf{h}}_3^2 + \mathbf{f}_2 \tilde{\mathbf{h}}_1^2 + \mathbf{f}_2 \tilde{\mathbf{h}}_1 \tilde{\mathbf{h}}_3 \\
& + \tilde{f}_1 h_2 h_3 + \tilde{f}_2 h_1 h_2 + \tilde{\mathbf{f}}_1 \mathbf{h}_2^2 + \tilde{\mathbf{f}}_2 \mathbf{h}_2^2 + \tilde{\mathbf{f}}_2 \mathbf{h}_2 \mathbf{h}_3 + \tilde{\mathbf{f}}_2 \mathbf{h}_3^2 ,
\end{aligned} \tag{A.24}$$

whereas for  $\langle C_2 \rangle \neq 0$  the following texture appears

$$\begin{aligned}
W_{\text{Yukawa}} = & h_1 f_1^2 + h_2 f_2^2 + h_3 f_1 f_2 + \mathbf{h}_1 \mathbf{f}_1 \mathbf{f}_2 + \mathbf{h}_1 \mathbf{f}_2^2 + \mathbf{h}_3 \mathbf{f}_2^2 \\
& + \tilde{h}_1 \tilde{f}_1^2 + \tilde{h}_2 \tilde{f}_2^2 + \tilde{h}_3 \tilde{f}_1 \tilde{f}_2 + \tilde{\mathbf{h}}_1 \tilde{\mathbf{f}}_1 \tilde{\mathbf{f}}_2 + \tilde{\mathbf{h}}_2 \tilde{\mathbf{f}}_1^2 + \tilde{\mathbf{h}}_2 \tilde{\mathbf{f}}_1 \tilde{\mathbf{f}}_2 + \tilde{\mathbf{h}}_3 \tilde{\mathbf{f}}_1^2 \\
& + f_1 \tilde{h}_2 \tilde{h}_3 + f_2 \tilde{h}_1 \tilde{h}_2 + f_2 \tilde{h}_3^2 + \mathbf{f}_1 \tilde{\mathbf{h}}_2^2 + \mathbf{f}_2 \tilde{\mathbf{h}}_2^2 + \mathbf{f}_2 \tilde{\mathbf{h}}_2 \tilde{\mathbf{h}}_3 \\
& + \tilde{f}_1 h_2 h_3 + \tilde{f}_2 h_1 h_2 + \tilde{\mathbf{f}}_1 \mathbf{h}_1^2 + \tilde{\mathbf{f}}_1 \mathbf{h}_1 \mathbf{h}_2 + \tilde{\mathbf{f}}_1 \mathbf{h}_3^2 + \tilde{\mathbf{f}}_1 \mathbf{h}_1 \mathbf{h}_3 + \tilde{\mathbf{f}}_2 \mathbf{h}_1^2 + \tilde{\mathbf{f}}_2 \mathbf{h}_1 \mathbf{h}_3 + \tilde{\mathbf{f}}_2 \mathbf{h}_3^2 .
\end{aligned} \tag{A.25}$$

These expressions are grouped by  $SU(5)$  products as in the previous case.

## A.4 Stability Wall Texture: A Three Family Mass Hierarchy

An important question in string model building is the following: is there a natural texture in a heterotic vacuum which leads, perturbatively, to a one heavy and two light families? To explore this issue, let us consider an  $SO(10)$  theory of the type described in case A.2.2 where  $\langle C_1 \rangle \neq 0$ . Choose the bundle so that the  $SO(10)$  non-singlet field content is two  $f_1$  fields, one  $f_2$  and one  $h_1$  only. This is a three generation model with one pair of Higgs doublets (inside  $h_1$ ). From (A.14), one sees that the only allowed perturbative Yukawa coupling of this vacuum is

$$W_{\text{Yukawa}} = h_1 f_2^2 . \tag{A.26}$$

That is, there is one massive third family and two light generations as desired. Therefore, one can expect one heavy family to arise naturally in some smooth compactifications of the heterotic string.

## B Equivariant structures and quotient manifolds

In this Appendix, we briefly outline the procedure for constructing a vector bundle on a manifold  $X/\Gamma$  for some discrete group  $\Gamma$ . This is intended to provide only a brief introduction to the construction. For a more detailed discussion of building equivariant structures and Wilson line symmetry breaking on non-simply connected Calabi-Yau manifolds see [18, 30] and [63] for useful numeric tools for these calculations.

To begin, we consider a rank 4 bundle,  $V$ , on the bicubic hypersurface in  $\mathbb{P}^3 \times \mathbb{P}^3$ ,

$$X = \left[ \begin{array}{c|c} \mathbb{P}^2 & 3 \\ \hline \mathbb{P}^2 & 3 \end{array} \right]^{2,83} \quad (\text{B.1})$$

If we denote the coordinates on  $\mathbb{P}^2 \times \mathbb{P}^2$  by  $\{x_i, y_i\}$ , where  $i = 0, 1, 2$ , then a freely acting  $\mathbb{Z}_3 \times \mathbb{Z}_3$  symmetry is generated by [64],

$$\begin{aligned} \mathbb{Z}_3^{(1)} &: x_k \rightarrow x_{k+1}, \quad y_k \rightarrow y_{k+1} \\ \mathbb{Z}_3^{(2)} &: x_k \rightarrow e^{\frac{2\pi i k}{3}} x_k, \quad y_k \rightarrow e^{-\frac{2\pi i k}{3}} y_k. \end{aligned} \quad (\text{B.2})$$

As shown in [64], the most general bi-degree  $\{3, 3\}$  polynomial invariant under the above symmetry is given by

$$p_{\{3,3\}} = A_1^{k,\pm} \sum_j x_j^2 x_{j\pm 1} y_{j+k}^2 y_{j+k\pm 1} + A_2^k \sum_j x_j^3 y_{j+k}^3 + A_3 x_1 x_2 x_3 \sum_j y_j^3 + A_4 y_1 y_2 y_3 \sum_j x_j^3 + A_5 x_1 x_2 x_3 y_1 y_2 y_3 \quad (\text{B.3})$$

where  $j, k = 0, 1, 2$  and there are a total of 12 free coefficients,  $A$ , above. We shall take these coefficients to be generic (i.e. random integers) in the following. With the choice of the invariant polynomial (B.3), we can define the smooth quotient manifold  $\hat{X} = X/G$ .

The quotient manifold is related to  $X$  via the natural projection map,  $q : X \rightarrow \hat{X}$ . Using this relationship, we note that any vector bundle  $\hat{V}$  on  $\hat{X}$  can be related to a vector bundle  $V$  on  $X$  via the pullback map,  $q^*$ . That is, for any bundle  $\hat{V}$  over  $\hat{X}$ ,

$$q^*(\hat{V}) \simeq V \quad (\text{B.4})$$

is an isomorphism for some bundle  $V$  on  $X$ . This pulled-back bundle,  $V$  is characterized by the geometric property of *equivariance*. A vector bundle on  $X$  will descend to a bundle  $\hat{V}$  on  $\hat{X}$  if for each element  $g \in G$ ,  $g : X \rightarrow X$ , there exists a bundle isomorphism  $\phi_g$ , satisfying two properties. First,  $\phi_g$  must cover the action of  $g$  on  $X$  such that the following diagram commutes

$$\begin{array}{ccccc} V & \xrightarrow{\phi_g} & V & & \\ \pi \downarrow & & \downarrow \pi & & \\ X & \xrightarrow{g} & X & & \end{array} \quad (\text{B.5})$$

and in addition,  $\phi_g$  must satisfy a so-called *co-cycle condition*, namely that for all  $g, h \in G$ ,

$$\phi_g \circ \phi_h = \phi_{gh}. \quad (\text{B.6})$$

The set of such isomorphisms  $\phi_g$  are referred to collectively as an *equivariant structure*. The morphisms  $\phi_g$  form a representation of the group that act on the bundle, the so-called lifting of (B.2) to  $V$ . In addition,  $\phi$  induces a representation of the group that acts on the cohomology  $H^i(X, V)$  and it is precisely the *invariant* elements of this cohomology (under the action inferred from  $\phi$ ) that descend to the quotient manifold  $X/G$ . We will return to this later. For now, we consider the description of the bundle  $V$  on  $X$ .

## B.1 The “upstairs” theory

For our current purposes, we shall choose the following equivariant bundle  $V$  on  $X$ . The bundle is defined by the following short exact sequence:

$$0 \rightarrow \mathcal{F}_1 \rightarrow V \rightarrow \mathcal{F}_2 \rightarrow 0 \quad (\text{B.7})$$

where

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{O}(0, 2)^{\oplus 3} \rightarrow \mathcal{O}(2, 2) \rightarrow 0 \quad (\text{B.8})$$

$$0 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{O}(1, -1)^{\oplus 3} \rightarrow \mathcal{O}(1, 1) \rightarrow 0. \quad (\text{B.9})$$

The bundle  $V$  is defined as an extension of  $\mathcal{F}_1$  by  $\mathcal{F}_2$ , where  $\mathcal{F}_i$  are rank 2 bundles defined by monad sequences. Since  $c_1(\mathcal{F}_1) = (-2, 4)$  and  $c_1(\mathcal{F}_2) = (2, -4)$ , we see that  $c_1(V) = 0$  and  $V$  is an  $SU(4)$  bundle.

To analyze the properties of the four-dimensional effective theory associated to  $\hat{V}$  (including the stability-wall induced textures in its Yukawa couplings) we must first consider the “upstairs” bundle  $V$  and use its properties to determine those of  $\hat{V}$ , “downstairs”. To begin then, the “upstairs” spectrum of  $V$  is given by

$$\begin{aligned} n_{16} &= h^1(X, V) = h^1(X, \mathcal{F}_1) + h^1(X, \mathcal{F}_2) = 9 + 18 = 27 \\ n_{\bar{16}} &= h^1(X, V^*) = 0 \\ n_{10} &= h^1(X, \wedge^2 V) = h^1(X, \wedge^2 \mathcal{F}_1) = 18 \end{aligned} \quad (\text{B.10})$$

A simple analysis along the lines of [16, 23], shows us that  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are both stable independently, but that  $\mathcal{F}_1 \in V$  itself destabilizes  $V$  in a part of its Kähler cone. Since  $c_1(\mathcal{F}_1) = (-2, 4)$ , we see that there is a stability wall when  $t^2/t^1 = 1 + \sqrt{3}$ . We find that the regions of stability for  $V$  on  $X$  are as shown in Figure 4.

At this stability wall, the poly-stable decomposition of  $V$  is as the direct sum of the two rank 2, bundles,  $V \rightarrow \mathcal{F}_1 \oplus \mathcal{F}_2$ . As we have argued in the previous sections, this supersymmetric decomposition of  $V$  changes the structure group of the bundle from  $SU(4)$  to  $S[U(2) \times U(2)]$  and hence, an extra  $U(1)$  appears in the low energy gauge symmetry. At this locus in moduli space, the visible matter fields, the **16**s and **10**s in (B.10), carry a charge under the enhanced  $U(1)$  as shown in Table 13 (the general case of such a decomposition is given in Table 10 in Appendix A).

## B.2 The “downstairs” theory

We turn now to the final, three generation theory on the quotient manifold  $\hat{X} = X/\mathbb{Z}_3 \times \mathbb{Z}_3$ . By quotienting the Calabi-Yau threefold in (B.1) by the discrete symmetry in (B.2), we form the manifold

$$X = \left[ \begin{array}{c|c} \mathbb{P}^2 & 3 \\ \mathbb{P}^2 & 3 \end{array} \right]^{2,11}_{/\mathbb{Z}_3 \times \mathbb{Z}_3} \quad (\text{B.11})$$

Representation	Field name	Cohomology	Multiplicity
$(\mathbf{1}, \mathbf{2}, \mathbf{2})_2$	$C_1$	$H^1(X, \mathcal{F}_1 \otimes \mathcal{F}_2^*)$	90
$(\mathbf{1}, \mathbf{3}, \mathbf{1})_0$	$\phi_1$	$H^1(X, \mathcal{F}_1 \otimes \mathcal{F}_1^*)$	9
$(\mathbf{1}, \mathbf{1}, \mathbf{3})_0$	$\phi_2$	$H^1(X, \mathcal{F}_2 \otimes \mathcal{F}_2^*)$	9
$(\mathbf{16}, \mathbf{2}, \mathbf{1})_1$	$f_1$	$H^1(X, \mathcal{F}_1)$	18
$(\mathbf{16}, \mathbf{1}, \mathbf{2})_{-1}$	$f_2$	$H^1(X, \mathcal{F}_2)$	9
$(\mathbf{10}, \mathbf{1}, \mathbf{1})_2$	$h_1$	$H^1(X, \wedge^2 \mathcal{F}_1)$	18

Table 13: The “upstairs” field content of the decomposed bundle  $V \rightarrow \mathcal{F}_1 \oplus \mathcal{F}_2$ .

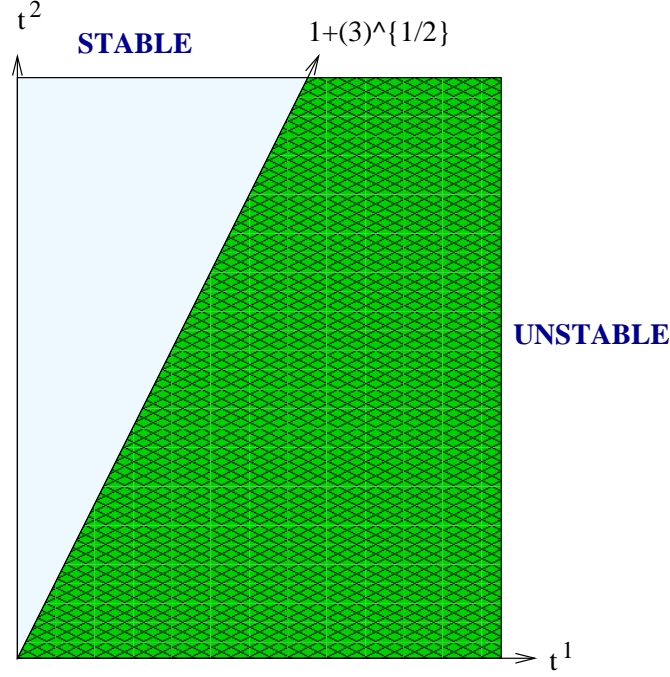


Figure 4: The Kähler cone ( $t^1, t^2 > 0$ ) and the regions of stability/instability for the “upstairs” bundle (B.7) on the simply connected Calabi-Yau (B.1). At the stability wall ( $t^2/t^1 = 1 + \sqrt{3}$ ),  $V$  decomposes as  $V \rightarrow \mathcal{F}_1 \oplus \mathcal{F}_2$  where  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are defined in (B.8).

Since each term in the short exact sequence (B.7) is equivariant, the entire sequence descends to a sequence of bundles over  $\hat{X}$ . We have

$$0 \rightarrow \hat{\mathcal{F}}_1 \rightarrow \hat{V} \rightarrow \hat{\mathcal{F}}_2 \rightarrow 0 \quad (\text{B.12})$$

Using the fact that the cohomology of  $\hat{V}$  is simply the invariant part of the cohomology  $V$  under the induced action of  $\phi_g$  in (B.5) and (B.6), we find that the number of **16**s and **10**s of  $SO(10)$  are

$$\begin{aligned}
f_1 : h^1(\hat{X}, \hat{\mathcal{F}}_2) &= 2 \\
f_2 : h^1(\hat{X}, \hat{\mathcal{F}}_1) &= 1 \\
h_1 : h^1(\hat{X}, \wedge^2 \hat{\mathcal{F}}_2) &= 2
\end{aligned} \quad (\text{B.13})$$



and that the charged bundle moduli  $C_1$  become

$$C_1 : h^1(\hat{X}, \hat{\mathcal{F}}_1 \times \hat{\mathcal{F}}_2^*) = 9 \quad (\text{B.14})$$

From the above, it is clear that we have produced a three generation  $SO(10)$  GUT theory. However, we can go still further by introducing Wilson lines which will break  $SO(10)$  to  $SU(3) \times SU(2) \times U(1)_Y \times U(1)_{B-L}$ . We shall not go into this breaking here, but refer the reader to [30, 29, 18] for details of breaking  $SO(10)$  with  $\mathbb{Z}_3 \times \mathbb{Z}_3$  Wilson lines.

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